Symmetry Lessons

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I created this unit for a summer program at New York's Museum of Math. This document is a slightly edited version of the unit, for use in elementary school classrooms, or in math circles. I suspect much of the material can be adapted for middle school and high school, since some of it originated (in a different form) in my high school **Space** course. (www.mathed.page/space)

This document includes lessons for both grades 1-3 and grades 4-6. Activities that I recommend for only one of the grade bands will be indicated explicitly by color and with a Greek letter: (*ε* for grades 1-3 and Δ for grades 4-6. The accompanying handouts and photos are all available at https://www.mathed.page/symmetry, along with links to other relevant materials.

The activities are based on my own experience in three ways:

- ◊ Ten years of teaching "enrichment" math in grades K-5, followed by occasional work with math circles at those levels
- ◊ Twenty years of teaching high school students about symmetry
- ◊ Thirty years as a curriculum developer and department chair (the latter role forcing me to think at a more granular level about how to implement curricular ideas as lesson plans.)

This is a good combination, but it does not guarantee infallibility. I encourage the teachers of these lessons to keep notes about how things turn out, so that whoever teaches this in future years learns from their experience. (Also, I welcome feedback!)

In addition to grade-level appropriateness, it is important to keep track of time: there may be more ideas in some lessons than can realistically be implemented in the time you have. The overflow can be saved for later use.

The lessons are organized as morning and afternoon sessions, Monday to Friday, but obviously that can be adapted to your schedule: just think of it as a ten-lesson unit.

Monday – Introduction to Symmetry
AM: Recognizing Symmetry
PM: Mirror puzzles

Tuesday – Special Examples
AM: Pentominoes
PM: Snowflakes

Wednesday – Beautiful Designs
AM: Kaleidoscopes
PM: Making Symmetric Designs

Thursday – Infinite Strips
AM: Frieze Symmetry
PM: Making Friezes

Friday – The Whole Plane
AM: Tiling
PM: Wallpaper Symmetry

Background

At the college level, symmetry is mostly approached from the point of view of group theory. This will not be the focus here, though it is certainly the underlying mathematics, especially in the second half of the week. I know from experience that some abstract algebra can certainly be injected into grades 1-6, but that is not the purpose of this week, which is mostly visual and geometric, so resist the pull in that direction. (I do have an abstract algebra unit, written for middle or high school, based on material I developed when teaching grades K-5: www.mathed.page/abs-alg).

The main focus of this unit is on developing a visual sense about the geometry of symmetric figures. This is achieved with a playful, hands-on approach, and an artistic component: students will see and create beautiful designs. Do not see this as taking anything away from the math. Quite the opposite: it is what makes the math interesting to many students.

Symmetric figures are figures that are invariant under isometries. Isometries are geometric transformations that preserve distance (and therefore angles.) The most familiar isometry is reflection in a vertical line. By the end of this unit, students should be comfortable with reflections in mirrors facing in any direction, and in fact in multiple mirrors. The other isometries are rotations, translations, and glide reflections. If you're not familiar with those, you may check out this Wikipedia page:

https://en.wikipedia.org/wiki/Euclidean_plane_isometry

or this page on my website:

https://www.mathed.page/transformations/

Again, avoid being drawn into a formal presentation of any of this. This background is for your own edification.

An excellent way to prepare to teach this unit is to spend some time with Peter Stevens' *Handbook of Regular Patterns* (MIT Press.) It is an extraordinary multicultural resource from which one can draw many examples. In it, he not only provides images from all over the world, but he also explains the math in detail for each symmetry group. However note that his notation is closer to that used by crystallographers than to standard mathematical notation. That discrepancy will not really affect this unit, since we will mostly not use notation — standard or otherwise. We'll get close to that when we use an ad hoc notation for the different frieze groups. But even there, we use it to throw light on the relevant symmetries, not to have students memorize or fully understand that classification.

Mathematically, the general sweep of the lessons is from finite symmetric designs (sometimes know as rosettes), to designs that are infinite in one dimension (friezes), and finally briefly to designs that are infinite in two dimensions (wallpapers.)

My *Geometry Labs* (free download on my website: https://www.mathed.page/geometry-labs) is an additional reference, though it is geared to somewhat older students. Some of the activities in this unit are derived from that book, and you can find more symmetry lessons in Section 5 therein.

The Education Development Center unit *Mirror Cards* (from the 1960's Elementary Science Study project) inspired this unit's mirror puzzles.

Materials

- Photocopied handouts (**handouts.pdf** and **make.pdf** from www.mathed.page/symmetry)
- Blank transparencies
- Markers that will work on transparencies
- Envelopes
- Mirrors (those are explicitly mentioned on Monday and Wednesday, but they should be available throughout the week. On Wednesday, you will need *two mirrors per student*.)
- Pencils
- Colored pens or pencils
- Interlocking cubes or square tiles
- Pentominoes (If you have access to a laser cutter, you can make your own using this file: https://www.mathed.page/puzzles/pento-labs/pentominoes.svg)
- Scissors
- Manila folders
- Adhesive tape
- Pattern blocks (lots)
- Geometry Labs Drawing Template (one per student) https://www.enasco.com/p/Geometry-Labs-Drawing-Template%2BTB18872
- Unlined paper
- Real graph paper (not photocopied) can be used instead of the 1cm grid paper in the handouts
- Colored paper (multiple colors, preferably construction or origami paper, not copier paper which is too pale)
- Glue sticks

The lessons assume that the teacher has access to a whiteboard and can project a computer screen. Given the topic, and the extensive use of special papers and graphics-heavy handouts, it would work better if the teacher had access to a document camera or an interactive whiteboard.

Interaction Modes

Discussion
Whole-class discussions, led by the teacher. This is formal: raise your hand if you want to say something, listen when someone is speaking, and so on.

The other modes are informal. The teacher circulates and interacts.

Group
Students work individually, but they are encouraged to ask for help from their neighbors as needed, to offer help when asked, and in some cases to split the work.

Pairs
Students work in pairs.

Solo
Students work individually, usually on creative projects.

Monday Morning
Recognizing Symmetry

Materials

- Mirrors
- **Letters** (from handouts) on transparencies
- Markers that will work on transparencies
- Envelopes
- Handout: **Alphabet**
- Handout: **Recognizing Symmetries** (two pages)

Preparation

- [I recommend arranging the furniture so students are in groups of four, which makes for productive conversations. It also makes it easy to split groups into pairs if/when needed.]
- Place a pencil and a colored marker at each seat.
- The discussion of the symmetry of letters would be more accessible, especially to the ε's, if you print the **Letters** handout on transparencies. Cut the transparencies to separate the letters, and put a whole alphabet in an envelope for each student.
- Work through the **Recognizing Symmetries** handout so as to be ready to help students as needed.

Discussion

- Classroom rules: raising hands, following instructions, helping each other, mistakes are not a problem
- Introductions
- Idea of the theme and overview of week's topics – also assess the students' familiarity
  - "Raise your hand if you know what symmetry is."
  - Listen to each answer in turn. Accept all answers. No right or wrong.
  - "Today, we're going to talk about two kinds of symmetry: mirror symmetry, and turn symmetry. Here is an example of mirror symmetry."
  - Draw a capital A on the board.
  - "I know it is mirror symmetric because if I put a mirror right in the middle, I still see a capital A"
  - Remove the mirror and draw the line of symmetry.
  - "The line where the mirror was is called the line of symmetry"
  - "Mirror symmetry is also called flip symmetry."
  - Distribute the envelopes.
  - "Find the A in your envelope. Put it on the table. Flip it over, it's still an A. That means it is flip symmetric"
  - "Find the F in your envelope. Put it on the table. Flip it over, it's not an F. That means it is not flip symmetric."
  - "On your paper, draw a capital F. Try putting a mirror on it so it's still an F. It's not possible, because an F is not mirror symmetric."
  - "Flip symmetry and mirror symmetry are the same. They are also known as line symmetry."
  - "On your paper, draw a capital M, and draw its line of symmetry. "
  - Show a capital C. "Draw a capital C"
  - "Does this capital C have a line of symmetry?"

- If students disagree, or if everyone says no, say "We can use a mirror to figure this out!"
- Place a mirror vertically, creating an O.
- "Oops! This is no longer a C! Maybe it is not symmetric!"
- If no student corrects you, have a sudden idea:
- "What happens if I put the mirror this way?" (Horizontally across the middle.)
- "It's still a C! It *is* symmetric, and here's the line of symmetry!"
- Draw the line.
- Repeat with a capital X, ending up with two lines of symmetry.
- Repeat with a capital Z, ending up with no lines of symmetry.
- "The Z doesn't have mirror symmetry. It has turn symmetry. If I turn it upside down, it's still a Z." (Demonstrate, using a large Z on a piece of paper.)
- "Find the Z in your envelope. And check that it has turn symmetry, but no line symmetry. To keep track of the turn, you can put a colored dot at the top left of the Z, and see what happens to it."
- "Is there another letter which will look the same if I turn it upside down?"
- Discuss their suggestions.

Group

- Distribute the **Alphabet** handout.
  - As students work, individuals may need help. Review with any students who are confused how to use the mirror to determine mirror symmetry, and to turn a letter upside down to find out about turn symmetry. Resist the temptation to "explain" verbally, as students will have trouble understanding you. Focus on the physical act of using the mirror, or turning the paper upside down.
  - Encourage students to help each other, compare their answers, and discuss their disagreements. If a group is having trouble, help them. If more than one group is struggling, stop everyone for a whole-class discussion.
  - Some letters are ambiguous, and can be written symmetrically or not. If students are arguing about those, tell them they can write the letter accordingly in the areas where they belong. (For example, if the B's curve is smaller at the top, it belongs in the "no symmetry" area. If the curves are identical, it belongs in the "mirror symmetry only" area. So the B can be in one area, or the other, or both, depending on how you write it.)
  - If some students finish much earlier than others, have them work on ambiguous letters and on the lowercase letters until others catch up.

Discussion

- "There are two ways to put the Z down so it's a Z" (demonstrate)
  - "What about this star? How many ways can I put it down so it looks exactly the same, with two arms at the bottom and one at the top?
  - 
  - "We say the Z has 2-turn symmetry, and this star has 5-turn symmetry." [Standard terminology is 2-fold rotational symmetry, but this is extremely confusing to stu-

dents, because no folding is involved. There will be folding in a future session, but that will be about mirror symmetry, not rotational symmetry.]

  - "Can someone draw a shape that has 4-turn symmetry?" (If no one has an idea, show a plus sign, or a square on a piece of paper.) "Let's turn it 4 ways to check". (After each 90° turn, ask whether it looks as it did before.)
  - Δ: "Another way to think about it is that if the figure is unchanged after ½ of a full 360° turn, it has 2-turn symmetry. If it is unchanged after 1/3 of a full turn, it has 3-turn symmetry. And so on."
  - "If a figure needs to be turned all the way around (360°) it can be called 1-turn symmetry, since the figure can only be seen one way. Such a figure is not usually considered to be symmetric."

Group

- Distribute the **Recognizing Symmetries** handout
  - Go over the instructions, marking up some examples on the board. Note that two examples have been marked, including a "1" for a figure with no symmetry other than the trivial 0° (or 360°) rotation.
  - Encourage students to help each other, compare their answers, and discuss their disagreements. If a group is having trouble, help them. If more than one group is struggling, stop everyone for a whole-class discussion.
  - The symmetries for many of the images are obvious, but some are subtle. Be prepared to discuss those with individuals, or groups, or the whole class.
  - ε: If some students are not able to grasp the more difficult cases, it is not necessary to fully settle those discussions. Tell students who insist on being told the right answer that you'll return to these in a few days when everyone is more used to symmetry.
  - The handout has enough examples, but if for some reason you want more, or different ones, you can find a huge number of images from every culture in the world in the Stevens book, pp. 2-92. (See the reference in the Background, on page 3 of this document.

Pairs

- Call two volunteers to the front.
  - "We will play mirror. You will be Player 1, and you will be Player 2."
  - "Player 1 will move slowly, and Player 2 will be their mirror image."
  - "For example, Player 1, raise one hand slowly. And Player 2 mirror that."
  - "The goal is to work together so that it looks like Player 2 really is the mirror image."
  - "What would happen if Player 1 moves suddenly or fast?"
  - "That's right. That's why the movements should be slow."
  - "After a while, I'll say 'Switch!'. At that point, Player 2 will be the person, and Player 1 will be the mirror image."
  - "Everyone, stand with a partner, and start!"
  - If there is an odd number of students, team up with one of them.
- Δ: "Everyone, stop!"
  - "Player 2: raise your right hand slowly, and Player 1: be the mirror image."
  - "Player 1: did you raise your right hand, or your left hand?"
  - "Let's see what happens if both of you raise a right hand slowly at the same time. Go!"

  - "This is 2-turn symmetry!" (Demonstrate by turning a student whose right hand is up to the other side.)
  - "Try to do the 2-turn game the same way as you played the mirror game. Move slowly!"
- Δ: [if there is time for this] "Go back to your table, and look at your Alphabet paper. Can you make symmetric words?"
  - Show some examples: BOX has a horizontal line of symmetry. SOS has 2-turn symmetry. WOW (if written vertically, with each letter beneath the preceding one) has a vertical line of symmetry.

Monday Afternoon
Mirror Puzzles

Materials

- Mirrors
- Handout: **Mirror Originals**
- Handout: **Mirror Puzzles** (three pages, respectively for Δ and ε — print extras of both in case some students want easier or harder puzzles)

Preparation

- Set up the laptop so you can show photos on the TV.
- Work through the puzzles so as to be ready to help students who need it.

Discussion

- "This morning , we learned about two kinds of symmetry. Let's see what you remember. I will show you some images. For each one, talk to others in your group and decide what symmetry you see in the photo. After you do that, I will pick someone to share their answer with all of us."
- "For each image, tell us: is there a line of symmetry? Is there more than one? What is the turn symmetry number?"
- Show photos from **On the Street** (www.mathed.page/symmetry/street.html). It is not necessary to go through all of them. In fact, it's best to reserve some to close today's session, and/or to open or close future sessions.
- If there are disagreements, either within a group, or between groups, have the students explain their views and try to convince each other. This is more challenging than in the morning session, since it will not be possible to use a mirror, or to turn the images around.

Discussion

- Distribute the handouts. Students will place the mirrors on **Mirror Originals** in order to solve the puzzles on **Mirror Puzzles**, where they will record their solution.
- The instructions are on the handout, but do not expect students to readily understand those. You might start by having students read the instructions silently, and then have someone read them aloud.
- Explain the challenge, using the first two examples. One is already done, in that the mirror line and the side one looks from are already indicated. Still, make sure every student is able to place the mirror correctly, to get the smiley face, and to understand why the arrow is on the left. The second example is a good check of whether they understand. Walk around to see that everyone gets it. If someone does not understand the nature of the puzzles, they cannot participate meaningfully.

Group

- [This activity does not include rotational symmetry. In it, we zero in on mirror symmetry. Also note that working with an actual mirror is different from mathematical reflection in a line. The reason is that the mirror only reflects items that lie on one side of the line. Mathematical reflection works on both sides simultaneously.]
- If some Δ's find their puzzles too difficult, you can give them the ε version, and vice versa.
- Note that some puzzles cannot be solved: perhaps the target is not symmetric, or it is symmetric, but cannot be obtained with a physical mirror, or it is rotationally symmetric, but

not line symmetric. This is intended to trigger conversations. Students can write "IMPOSSIBLE" next to those.

- Students may declare some other puzzles are impossible, when in fact they are just difficult. Do not contradict them. Instead, make that a discussion topic.
- If some students finish early, encourage them to make mirror puzzles based on their own drawings. If everyone finishes early, you can go back to the photos.

Tuesday Morning
Pentominoes

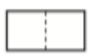
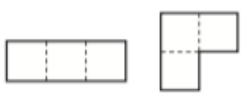
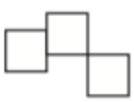
Materials

- Handout: **One-centimeter grid paper**
- Interlocking cubes or square tiles
- Pentominoes
- Handout: **Cover with Pentominoes** (two pages), and **Pentomino Puzzles**. They should be printed one-sided! Make sure the sides of the figures are multiples of one inch — adjust the copier settings if needed)
- Handout: **One-inch graph paper** (Again, make sure the sides of the squares measure one inch — adjust the copier settings if needed)

Preparation

- Make sure each pentomino set is complete, and that you have sets in different colors.
- New groups: randomly assign tables to students, for example by handing out playing cards as they come in. (Doing this every morning helps to keep the focus on the lesson, rather than on the inevitable socializing.)

Discussion

- "What does a domino look like?"
- Draw one 
- "It is made up of two squares, joined edge-to-edge."
- "If I use three squares, joined edge-to-edge I can make two different shapes! They are called trominoes."
- 
- "This is not a tromino, because the squares are not joined edge-to-edge:
- 
- Hand out interlocking cubes or square tiles. [The advantage of interlocking cubes is that they allow one to rotate or flip over the resulting shapes, which helps eliminate non-obvious duplicates. The disadvantage is that they make it possible to find 3D shapes which are not relevant to today's activity. To rule that out, insist that all cubes must touch the table. The advantage of square tiles is that they are a better model of what we are looking for, but the resulting figures cannot be manipulated easily.]
- "Use these to make tetrominoes. Those are made of four squares, joined edge-to-edge. Try to find different shapes."
- Walk around and encourage students to find all the shapes. There are five different ones. Gradually copy shapes found by the students onto the board. If students find shapes that are already on the board, but in a different orientation, tell them they are not considered different in today's activity.
- When all five tetrominoes have been found:
- "We have found all five tetrominoes. Now find the pentominoes. They are made of five squares, joined edge-to-edge."
- Hand out the one-centimeter grid paper.

- "Keep track of the pentominoes you find by drawing them on the grid paper."
- There are 12 pentominoes. It is not necessary for all students to find all of them, or for you to draw them on the board. When most students have found many different pentominoes, you can hand out the plastic pentominoes. Make sure neighbors get different colors, so that the sets don't get mixed up.
- "Arrange your pentominoes in four piles: mirror symmetric only, turn symmetric only, both kinds of symmetry, neither."
- Make sure everyone agrees on which pentomino belongs in each category.

Group

- Distribute the **Cover with Pentominoes** handouts.
- "You will use your pentominoes to cover symmetric shapes."
- "Each shape can be covered with two pentominoes."
- Δ: "An extra challenge is to cover them all at the same time with the pentominoes from one set."
- "Let me know when you're ready for three-pentomino symmetric puzzles"
- Distribute **Pentomino Puzzles**.
- "Each figure can be covered with three pentominoes in many ways. How many can you find? Keep track of your answers by writing the names of the pentominoes you used next to the shape."
- Δ: Distribute one-inch grid paper.
- Δ: "After you've found many answers to the three-pentomino puzzles, try to make a symmetric shape using four pentominoes. If you find one, trace it on the one-inch grid paper."

Make sure you leave some time to collect the pentominoes. Remind students that twelve pieces must find their way into each bag!

Tuesday Afternoon
Snowflakes

Materials

- Unlined paper (cut in half: 5.5” by 8.5”)
- Scissors
- Rulers
- Manila folders
- Handout: **Cut and Fold**
- Handout: **Fold and Cut**

Preparation

- Make extra copies of **Cut and Fold** so as to be ready if some students “mess up”.
- Make sure the scissors you’ll be using can cut through multiple layers of paper. Some “safety” scissors may not work for this activity.
- Do #3 of **Fold and Cut** for practice, so you can be helpful if students find this challenging.
- Do the Δ activity at the very end of the lesson yourself, for the same reason.

Group

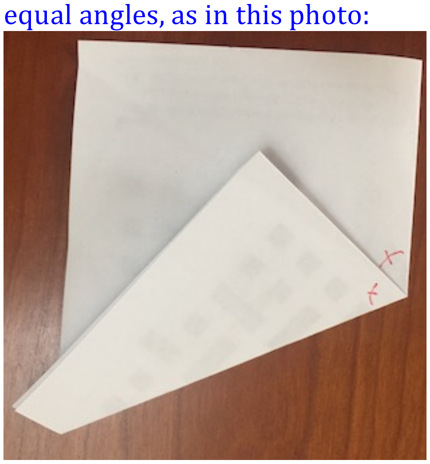
- Distribute the **Cut and Fold** handouts, the scissors, and the rulers.
- “Line symmetry has other names. Do you remember them?” (flip symmetry, mirror symmetry)
- “Can you explain those names?”
- “Line symmetry has another name: fold symmetry. That is what we will explore this afternoon.”
- The instructions are
    - cut out each shape
    - fold it in half and unfold it to show the line of symmetry
    - do it again until you have shown all the lines of symmetry for each figure (2, 3, 4, 5, and 6)
- Students who don’t want to (or can’t) deal with the complexity of the general’s insignia can use a ruler to draw lines joining the outermost vertices. The resulting pentagon will be easy to cut out and fold. In fact, it may be best to suggest this to everyone especially the ε’s, but allow Δ’s who are up to the challenge to cut out the figure.

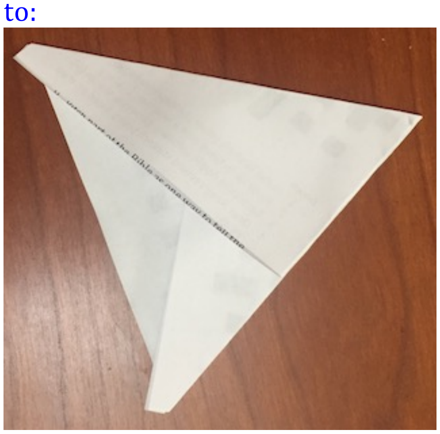
Group

- Distribute the **Fold and Cut** handouts.
- Encourage students to use a ruler to draw the fold lines. Even though the papers were folded no more than three times, there are four fold lines in the third and fourth figure.
- We have 1-turn symmetry, 2-turn symmetry twice, and 4-turn symmetry. The third figure prefigures frieze symmetries, as the right side is a translation of the left side. (Do not bring this up! If a student says something along these lines acknowledge it is a valid observation.)
- Distribute the unlined paper. Give each student several sheets.
- #3: Making the given designs will require making the right folds, and cutting in the right places. This will be challenging for some. Be prepared to help, and in the case of the ε’s, allow students who find this too hard to just go on to #4.
- As they make their designs, circulate and discuss the symmetry of each figure.
- For #4, do not let students make more than four folds. As they work, make sure they make each fold divide the visible surface in half. Other folds are possible, and perhaps interesting,

but there would not be time to fully explore the implications.
(For more challenging fold-and-cut puzzles, see Katherine Paur's "Cutting Out Hearts": https://www.mathed.page/puzzles/hearts.pdf)

- Ideally, you should have a bulletin board or chart paper display of the resulting figures, with examples of all four (or more) types of symmetry. If you will be doing that, students should write their name on their creations. Make sure to post equal numbers from each student. This will look best if the background is not white.
- Give students manila folders so they can take their creations home. Alternately, have them write their name on the folders, and tell them they'll be able to add more things to the folder in the next few days. In that scenario, you'll keep the labeled folders at the museum until Friday.
- Δ: If some students are up for the challenge, they can try to make folds that would yield three-turn symmetry. They are not likely to figure this out on their own, so if they get frustrated, suggest this way to do it:
  - Make the first fold down the middle as in the previous examples.
  - Make the second fold from the middle of the first fold, so as to get approximately equal angles, as in this photo:

    
  - Make the third fold so that the angles marked above meet each other, as in this photo:

    You might need to adjust the second fold to make everything come out right.
  - Make one or more cuts, and unfold!

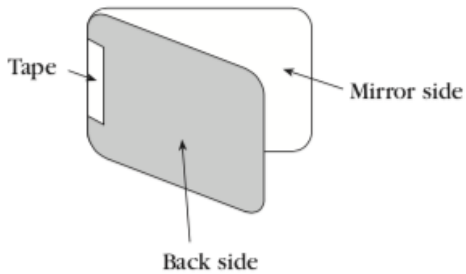
Wednesday Morning
Kaleidoscopes

Materials

- Pattern blocks
- Two mirrors per student
- Handout: **Dodecagons** (Make sure the sides of the dodecagons measure one inch and two inches — adjust the copier settings if needed)

Preparation

- Tape mirrors together in pairs:



- Set out pattern blocks on all the tables
- New groups: randomly assign tables to students, for example by handing out playing cards as they come in.

Discussion

- Give students a few minutes of free play with the pattern blocks. If you see symmetric creations, point them out, ask the creator to identify the symmetry, and take a photo for possible sharing on the TV in the afternoon.
- Δ: If some students have an understanding of angle measure, you can challenge them to figure out the angles on the pattern blocks.
- Distribute the hinged mirrors.
- "Put one or two pattern blocks between the mirrors. Looking into the mirrors, you'll see copies of your blocks, making a design. Experiment! What happens when you change the angle between the mirrors?"
- Give students a few minutes to experiment.
- "Now move all the blocks out from between the mirrors, and put in one blue rhombus. Change the angle between the mirrors and the position of the block so you can see 3 copies of it (including the original). 4 copies, 5, 6. Is 7 possible?" (Yes, but it's not easy.)
- Δ: "What is the angle between the mirrors when we see 3 copies? 4?" (etc.) The easiest case is that we need a 90° angle to see 4 copies (including the original.) If students have trouble figuring out the other angles, point out that 360°/4 = 90°. This should lead to a strategy.
- "Repeat this with other blocks. 3 copies? 4?" (etc.)
- Δ: "Do we get different angles for different blocks?"

Solo

- Distribute the **Dodecagons** handout. Collect the mirrors.
- "These figures are called 'dodecagons'. The name comes from the Greek: 'dodeca' means 12, and 'gon' means angle. These figures have 12 angles and 12 sides."

- "The figures can be covered with pattern blocks. You can choose whether to work on the small dodecagon, or the large one."
- "As a whole group, we'll try to find all these symmetries."
- Write on the board:
  1-turn no mirror, 1-turn with mirror
  2-turn no mirror, 2-turn with mirror
  3-turn no mirror, 3-turn with mirror
  4-turn no mirror, 4-turn with mirror
  6-turn no mirror, 6-turn with mirror
  12-turn no mirror, 12-turn with mirror
- "All these are possible! I'll put a check mark on the board when that symmetry has been found, and I'll take a photo of that dodecagon." (Note that students typically find the no-mirror solutions much more difficult to create. When someone succeeds, you may encourage others to take a look at how it was done.)
- Δ: Extra pattern block dodecagon challenges can be found here: www.mathed.page/manipulatives/pattern-blocks/dodecagons

Wednesday Afternoon
Creating Symmetric Designs

Materials

- Colored pens or pencils for each table
- Geometry Labs Drawing Templates
- Handouts: **Make a Design** (make extra copies, as students may "mess up".)
  ε's: pp. 1-10. Δ: pp. 1-16.

Preparation

- Familiarize yourself with the template.
- Try the "**Challenge**" pages of the handout, so as to be ready to help students who may miss some required reflections.

Discussion

- Distribute the templates, and give students some free time to make any designs they want, symmetric or not.
- "Some of the shapes in your template are symmetric. Trace those, draw their lines of symmetry, and write the number of their turn symmetry."
- Note that one cannot draw all the circle's lines of symmetry, as any line through its center works. Also any rotation around the center works.
- "On a blank piece of paper, draw a line down the middle. Choose a shape that is not symmetric, and trace it on one side of the line. Now use the template to draw its reflection on the other side." (To do this right, students have to flip the template over! Make sure everyone understands that.)
- Δ: "Some of the shapes in your template are called *regular polygons*. What that means is that they have straight sides, all the sides are equal, and all angles are equal."
- "Find those shapes. Let's make a list of how many sides they each have." (Make a list on the board. A possible misconception would be to classify non-square rhombuses as regular polygons. Among quadrilaterals, only the square is regular. On the template are 3, 4, 5, 6, 7, 8, 10, and 12-gons.)
- "For some regular polygons, the mirror lines go through opposite corners, or through the middle of opposite sides. For other regular polygons, the same mirror line connects the middle of a side with the opposite corner. Is there a pattern to that?" (The latter are the odd cases: 3, 5, and 7. The former are the even cases: 4, 6, 8, 10, and 12.)

Solo or Pairs

- "This afternoon, you will use everything you learned so far to make your own symmetric designs. You will each choose your best design or designs to post on our group (bulletin board? Online?). Be sure to put your name on your designs!"
- Distribute the **Make a Design** handout.
- "Look through the handout. Some pages have a square grid, some have a triangle grid, and some don't have any grid." (Hold up examples of each.)
- "All the pages have lines of symmetry. To make your design, you start by drawing or coloring something. Your original drawing should not cross the lines of symmetry. If you're using the grid, use the grid lines as guides for your original figure. If you're not using the grid, use your template to draw something."

- “Then draw the reflection of your drawing across all the lines of symmetry. But that is not all! You also need to draw the reflection of the reflections, and so on, until there are no more to draw. Then you can add a new original drawing, its reflections, their reflections, etc.”
- Demonstrate this on the board.
- “You can work by yourself or with a partner, and make your designs on a page of your choice.”
- (One way to encourage a diversity of designs is to try to see if students will volunteer to take on different pages. You can make a list of the pages on the board, and write names of volunteers. Another way is to draw assignments randomly from a hat, then allow students to trade their assignments as they see fit. In most groups, that works, as some students prefer more challenging tasks, and others don’t. In any case, do not force anyone to do something they don’t want to do.)
- Δ: “Note that the pages marked “**Challenge!**” are more difficult! After you’ve completed one, look for additional lines of symmetry. How are they related to the original lines of symmetry?” (They are reflections of the given lines.)
- Δ: “If you want to make a design with rotation symmetry and no line symmetry, you can still use the handout, but instead of reflecting your original shape, use the lines to guide you as you rotate it.” (Demonstrate on the board.)

Thursday Morning
Frieze Symmetry

Materials

- Graph paper
- **Friezes** handout
- **Frieze Patterns** handout (Δ's only)

Preparation

- Work through the **Friezes** activity, to make sure you're ready to help students.
- New groups: randomly assign tables to students, for example by handing out playing cards as they come in.

Discussion

- "What are the two types of symmetry we have learned about?" (line symmetry, turn symmetry)
- "Today, we learn about two other types, that we can see only in infinite designs."
- Draw a sequence of L's on the board, like this: … L L L L L …
- "Imagine these L's go on for ever in both directions. An infinite pattern like that is called a *frieze*" (write the word on the board)
- "If I move all of them to the left just the right distance, you would not know I moved them."
- "This kind of symmetry is called *translation*. I will represent it with this arrow."
- … L L L L L …

  ← translation

  (the arrow points to the left, its length is the distance between two consecutive L's)
- "Can you think of other possible arrows I could have used?" (The arrow can point to the right. The length of the arrow can be any integer multiple of the original arrow.)
- Distribute graph paper.
- "On your paper, make a frieze of letters, and show some of the possible translation arrows."
- Circulate to make sure everyone understands this.
- "A frieze can have other symmetries, besides translations. Here is an example." (On a piece of paper, write:

  L ⅂ L ⅂ L ⅂
-
- "If I turn this upside down, it will look the same." (Demonstrate.)
- "L, upside-down L, L, upside-down L, and so on. So it has 2-turn symmetry, and I can write a 2 next to it." (Do it.)
- "It does have translation symmetry also, but be careful: the arrow must go from an L to another L, or from an upside-down L to another upside-down L." (Demonstrate on the board.)
- "On your paper, make a frieze made of capital L's. They can be normal L's, backwards L's, upside-down L's, or both backwards and upside-down L's." (Show all four versions on the board.)

  L ⅃

  Γ ⅂
- "You can use one or two rows. Make sure your pattern repeats, and has translation symmetry."

- Circulate, looking for an example that has vertical mirror symmetry, and an example that has horizontal mirror symmetry, and copy them on the board. (Make sure they have translation symmetry!) If no one came up with any, just put your own examples on the board, using these possibilities:
- Vertical mirrors:

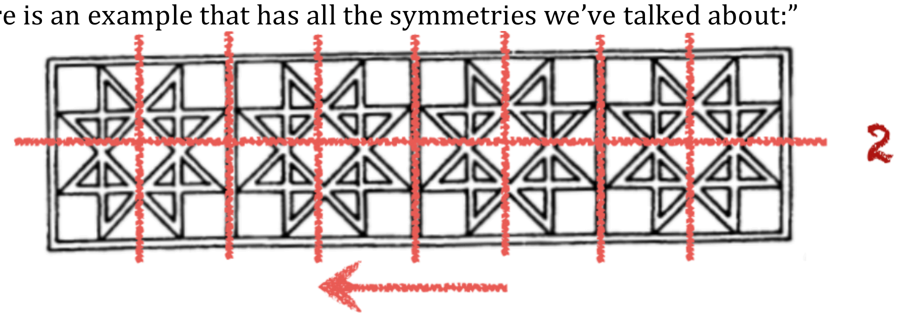
  Horizontal mirror:

- Draw the mirror lines on these examples. (Note that in the case of the vertical mirrors, they land in two sorts of locations: half are “in front” of the L’s, and half are “behind” the L’s.)
- (Note also that because of translation symmetry, if there is one vertical mirror, there are an infinite number of vertical mirrors. In contrast, because the frieze has finite “height”, there can only be one horizontal mirror across the middle of it. It is unchanged by the translations.)

Group

- Distribute the **Friezes** handout.
- “For each frieze:
  - Draw a translation arrow. All these friezes have translation symmetry.
  - If it has 2-turn symmetry, write a 2 next to it.
  - If it has line symmetry, draw the line or lines in color.”
- “Here is an example that has all the symmetries we’ve talked about:”

  (Display this on the TV.)
- “If you are not sure about the symmetries on a certain frieze, talk to your neighbors to sort it out! Definitely compare your ideas at the end of each page.”
- Circulate. Do not expect the students to find all the symmetries in a first pass. (This is especially true of the ε’s. Students will be stumped by the examples that featured glide reflections without a horizontal mirror. If the question comes up, tell them that there is a two-step symmetry: translation then reflection, or reflection then translation.)

Discussion

- After a while, discuss the friezes one by one as a whole class, soliciting ideas from the students. Hopefully the group will find all the rotations and reflections by the end of the discussion, but in any case, they will learn from it.
- “Friezes can have one more kind of symmetry, called the *glide reflection*. Here is an example:

...L Γ L Γ L Γ L Γ L Γ L Γ...

To get from one element to the next, we flip it over, and then translate (or translate, and then flip it over.) Here is a way to show the symmetries:”

- “The big arrow represents the translation symmetry. The dotted line and the small curved arrow show the glide reflection.”
- (For more information about glide reflections, see: https://www.mathed.page/transformations/isometries/glide-ref)

Group

- Δ: Distribute the **Frieze Patterns** handout. (There are lots of different notations for the seven groups — six on Wikipedia, and another one in Stevens’ book. All are difficult to parse and remember, even Conway’s cute idea involving feet. The notation on the handout is an attempt at something elementary school students and other non-specialists can readily understand and remember. Note that the “G” is not mentioned if there is an “H” in the name, as it is automatically implied by the fact we have a reflection line and a parallel translation. None of this needs to be discussed with the students!)
- “Mathematicians have proved that there are only seven frieze patterns. Go through the friezes on your handout and label them based on their symmetries: T, TV, TH, T2, TG, T2HV, or T2VG.”
- Circulate and help students.

Thursday Afternoon
Making Friezes

Materials

- **Frieze photos** (www.mathed.page/symmetry/friezes.html)
- **Make a Frieze** handout
- **Frieze Patterns** handout
- Unlined paper
- Drawing template
- Colored paper
- Colored pens or pencils
- Scissors
- Glue sticks

Preparation

- Familiarize yourself with the photos, so as to be sure to catch all the symmetries when sharing them with the class.
- Set up the computer to be ready to project the photos.

Discussion

- "This morning, we learned about friezes. What is a frieze?" (A possible definition is: an infinite strip with a repeating pattern. Accept all student answers, while directing them to two key ideas: the design is infinite in one dimension, and it includes at least translation symmetry.)
- "I will show you some friezes, and we'll try to find all their symmetries." (Show the frieze photos on the TV and discuss their symmetries. Δ: If you think students are ready for this, discuss which pattern applies to each image. If you didn't have time to introduce the patterns in the morning, you can do it now.)

Solo

- "This afternoon, you will make your own frieze designs. You can use the **Frieze Patterns** handout for ideas of the different possibilities. On that sheet, the basic element is a comma (or apostrophe), but of course you can use any shapes you want."
- "Your designs will only look good if the shapes you are repeating are exactly or almost exactly the same. There are three ways to do that:
    - Make shapes out of colored paper, cutting several copies at the same time — if you do that, you will use glue sticks to paste down your shapes.
    - Work on grid paper, and use the grid paper lines as guides.
    - Use the drawing template."
- Distribute the handouts and materials.
- For each design, allow students to choose their background (colored paper, unlined paper, grid paper) and the way they will make their figures (the three methods described above.) The one combination that seems problematic is using the template on the grid paper.
- As students work on their designs, discuss the symmetries with them.
- All designs must have translation symmetry! Therefore the repeating figures must be spaced evenly. The rulers on the drawing template can help if one is not working on grid paper.
- If there are vertical mirror lines, they too must be spaced evenly!

- Δ: If you want to aim for all patterns to be represented, you can nudge students towards the missing patterns. Or you can have students draw patterns out of a hat, and trade with each other if they're not happy with what they got. (If someone cannot find a trade, allow them to just choose whatever pattern they want.)

Friday Morning
Tiling

Materials

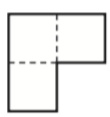
- Grid paper (the **One-cm paper** from the handouts, or real graph paper)
- **Triangle paper**
- Unlined paper
- Drawing Template
- Colored pens or pencils

Preparation

- You might read up on Archimedean tilings (see for example Wikipedia)
- New groups: randomly assign tables to students, for example by handing out playing cards as they come in.

Discussion

- Distribute grid paper and triangle paper
- "Imagine you had an infinite supply of tiles, for example square tiles. You could cover an infinite plane with them. (In math, a plane is an infinite flat surface.)"
- Demonstrate on the white board.
- "This morning you will explore tiles of different shapes. Let's start on the grid paper, using this tile:"



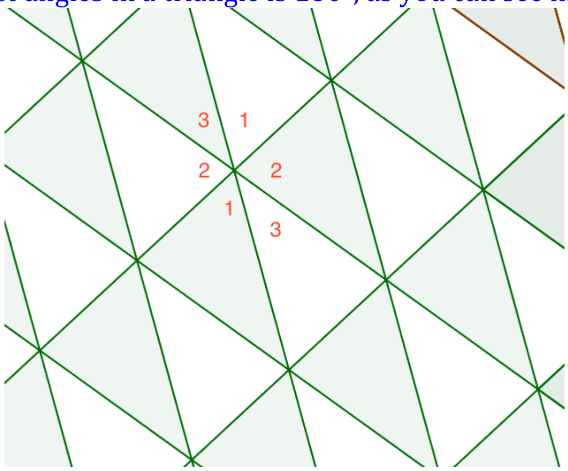
- "Can you tile the whole plane with it?"
- Circulate to see the students' work. There are many ways to do this, so you might hold up interesting variations.
- "How can we be sure we have a system that could go on forever?" (One way is to show that you can continue a strip in both directions, and that you can juxtapose two such strips. Demonstrate on the board, using the above tile.)
- "Now choose your own tile, made up of no more than five squares or triangles, and see if you can tile with it. You don't have to fill the whole page: just get far enough to be sure that you can go on forever in all directions. Once you're sure, you can color the design, or find another tiling."
- As students work, hold up successful examples for everyone to see. As you circulate, help students figure out whether their design will work. Can they extend a strip? Can they juxtapose strips?

Solo

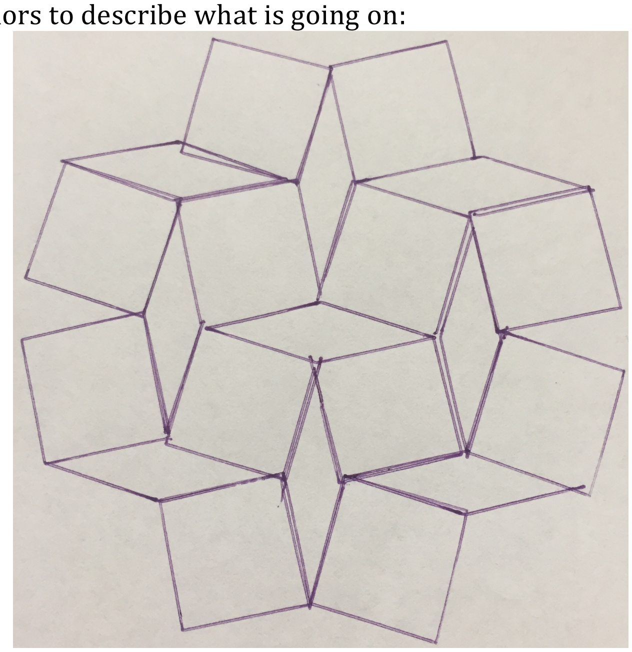
- Distribute unlined paper and drawing templates.
- "Now choose a triangle on the template, different from your neighbors', and tile with it."
- Circulate. A helpful hint for students who are stuck is to match equal sides. All triangles will tile, with no need to flip the template over. For students who are seriously stuck, a bigger hint is that turning the template 180°, but not flipping it over, can be helpful.
- Again, once students are sure their scheme will work they can color the tiling.
- Repeat with quadrilaterals. "A quadrilateral is a shape with four straight sides. You have many of those on your template." This is more challenging, but all quadrilaterals will tile,

even the non-convex example on the template. The same hints apply as in the case of triangles.

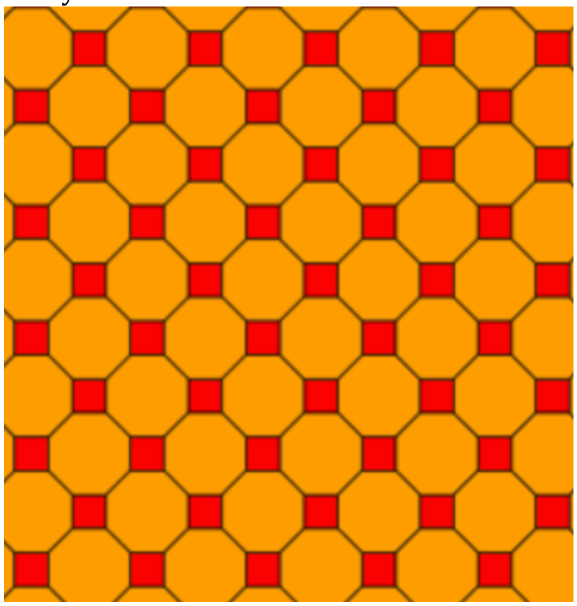
- Δ: If some students have an understanding of angle measure, you can discuss with them the sum of angles at vertices in the tiling. Many triangle and quadrilateral tilings can be used to show that the sum of angles in a triangle is 180°, as you can see in this example.



- Using quadrilateral tilings, a similar argument can be made about the sum of angles in a quadrilateral.
- "Now let's make tilings using any two tiles from your template, but of course not including the circle. Also, do not use the triangles or quadrilaterals, except for the ones with all equal sides: the green, orange, blue and tan pattern blocks." (This will make it possible to combine two different tiles, which would not be possible if using other triangles or quadrilaterals.) "All your tilings should be edge-to-edge."
- To save time and frustration, you should discourage the use of the 5-, 7-, and 10-gons. With more time, and more mathematical maturity, it would be possible to show why those will not work, but that is not an option in this session.
- A good hint is that for a pattern to extend infinitely, it is helpful to keep vertices consistent. For example, in this pattern, all vertices are surrounded by "orange, tan, orange, tan" using the pattern block colors to describe what is going on:



- Likewise, patterns involving regular polygons can be described as (for example) “4, 8, 8”, working our way around any vertex.



(image from Wikipedia)

- Δ: Once a tiling has been found, students who understand angle measure can figure out the angles. For example, in the example above, the octagon angle can be figured out: it is half of 360 – 90 degrees.
- Δ: If someone wants to use three tiles, warn them that this will be more difficult. The only reason to allow it is that it makes it possible to come across all the Archimedean tilings. However, it’s not likely all will be found in such a short time, and it’s not particularly crucial, so it is probably best to just limit students to two tiles.

Friday Afternoon
Wallpaper Symmetry

Materials

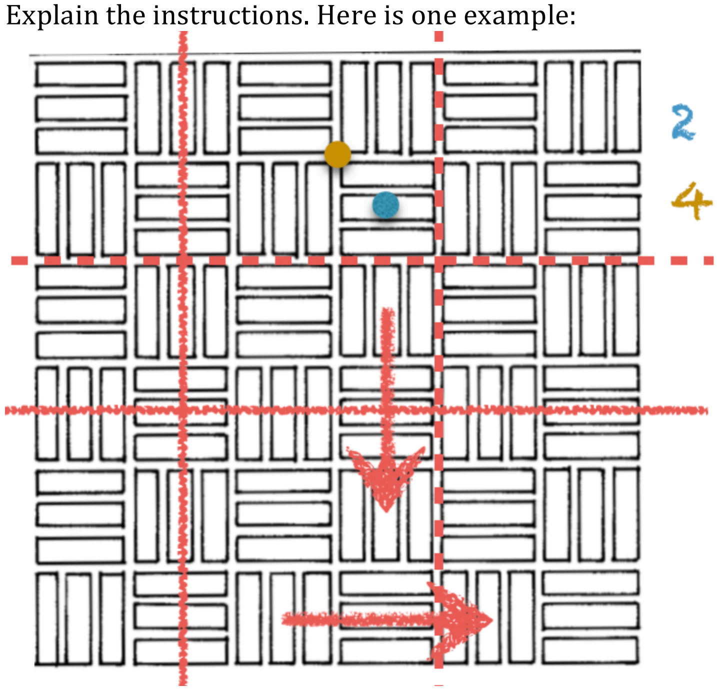
- **Wallpapers** handout (five pages)
- **Make a Wallpaper** handout (14 pages, but you won't need the whole set for each student. Maybe make half as many sets as you have students.)
- Drawing template
- Colored pens or pencils
- Mirrors should be available to students who request them.

Preparation

- Do the **Wallpapers** activity so as to be ready to help students
- Lay out the **Make a Wallpaper** handouts to allow students to choose the one(s) they will work on in the second half of the session.

Group

- Distribute the **Wallpapers** handout.
- Explain the instructions. Here is one example:



- (Not all possible translation vectors are shown. Opposites and combinations of these two also work. Likewise, only sample reflection and glide reflection lines are shown. Two sample rotation centers are shown, color-coded. Fortunately, many of the examples in the handout have fewer symmetries!)
- Even though students by now should be familiar with the different sorts of symmetries, finding them in a portion of an infinite pattern is actually quite difficult. Do not expect students to find them all!
- Choose some examples to discuss as a whole class.

Solo or Pairs

- "You will work by yourself or with a partner to make wallpaper designs, using the lines of reflection as your guides."
- "Each paper has lines of reflection, and a starting design element, usually a triangle. Some are on grid paper or triangle paper, others are on plain white paper. If you are working on plain white paper, you should use the drawing template."
- "You will choose which paper to work on, but first, let me explain the instructions for this activity."
- "1. Draw and color the reflections of the original design element in the given mirror lines. Draw and color the reflections of the reflections. Keep going until there are no more possibilities."
- "2. If there is room for that, create an additional design element. It should not cross the lines of reflection. Draw and color its reflections in the given mirror lines. Draw and color the reflections of the reflections. Keep going until there are no more possibilities."
- "Now choose your paper! (from **Make a Wallpaper**) When you're done with this one, you'll choose another one!"
- (One way to encourage a diversity of designs is to try to see if students will volunteer to take on different pages. You can make a list of the pages on the board, and write names of volunteers. Another way is to draw assignments randomly from a hat, and then allow students to trade their assignments as they see fit. In most groups, that works, as some students prefer more challenging tasks, and others don't. In any case, do not force anyone to do something they don't want to do.)