

Proportional Relationships: Connections

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A set of activities using multiple mathematical tools (and some “real world” applications) to help middle school students make connections between proportional relationships and other concepts, including area, scaling, slope, functions, and rate of change.

For related activities, teacher notes, and in some cases answer keys, go to
www.MathEducationPage.org

Investigating Rectangle Areas p. 1

This lesson is adapted from Lesson 1.A in *Algebra: Themes, Tools, Concepts*. The whole book can be found free on my Web site, along with a Teachers’ Edition and much in the way of support materials.

Scaling on the Geoboard, Similar Rectangles p. 3

These labs are excerpted from *Geometry Labs*. The whole book can be found free on my Web site. They are intended to be used with 11 by 11 geoboards, which can also be used for many other lessons, including work on area and the Pythagorean theorem. One implementation comes with a circle geoboard on the back, which is useful for an introduction to geometry and trigonometry. (The CircleTrig Geoboard is available from Nasco.com. More information is available on my Web site.)

In the absence of geoboards, you can do these labs using dot paper, also available on my Web site.

Discounts p. 6

Adapted from Lesson 6.A in *Algebra: Themes, Tools, Concepts*.

Guess My Function, Nine Function Diagrams, Sixteen Function Diagrams p. 7

These can be found in the Function Diagrams section of my Web site.

Constant Speed p. 11

Adapted from Lesson 4.1 in *Algebra: Themes, Tools, Concepts*.

In the Lab p. 13

Adapted from Lesson 4.6 in *Algebra: Themes, Tools, Concepts*.

Investigating Rectangle Areas

In this lesson, we will investigate rectangle areas by looking for patterns in tables and graphs. These are the questions we will be thinking about:

- ◇ How does the area of a rectangle change if you vary the length or width and leave the other dimension unchanged?
- ◇ How does the area of a rectangle change if you vary both the length and the width?

1. What is the area of a rectangle with the following dimensions?
 - a. 1 by 9
 - b. 2 by 9
 - c. 3 by 9
 - d. 9 by 9
2. What is the area of a rectangle with the following dimensions if $x = 10$?
 - a. 1 by x
 - b. 2 by x
 - c. 3 by x
 - d. x by x
3. Fill out this table:

Rectangle area	
x	1 by x
1	1
2	
3	
4	
5	
6	

4. Draw axes, with x on the horizontal axis, and area on the vertical axis. Plot the points you obtained in the previous exercise for the area of 1 by x rectangles. For example, (1,1) will be on the graph.
5. Does it make sense to connect the points you plotted? What would be the meaning of points on the line, in between the ones you got from your table?

6. Now fill out these tables:

Rectangle area	
x	2 by x
1	
2	4
3	
4	
5	
6	

Rectangle area	
x	3 by x
1	
2	
3	9
4	
5	
6	

Square area	
x	x by x
1	
2	
3	
4	16
5	
6	

7. On the same axes, graph the data you obtained for 2 by x, 3 by x, and x by x rectangles. For more accuracy on the last one, you may use your calculator to find points for $x = .5, 1.5$, and so on.
8. **Discussion:** Answer these questions about the graphs:
- Which ones represent a proportional relationship?
 - How do the first three graphs differ from each other? Explain.
 - What is special about the fourth graph?
 - Do the graphs intersect? What is the meaning of the intersections?
 - Where would the graphs meet the y-axis if we extended them?
 - Which area grows the fastest? why?
9. I am thinking of a number, which I will call k. If I made a table and a graph for rectangles with dimensions k by x, it would start like this:

Rectangle area	
x	k by x
1	5

What is k equal to? Explain.

10. Here is one part of a table:

Rectangle area	
x	9 by x
?	13.5

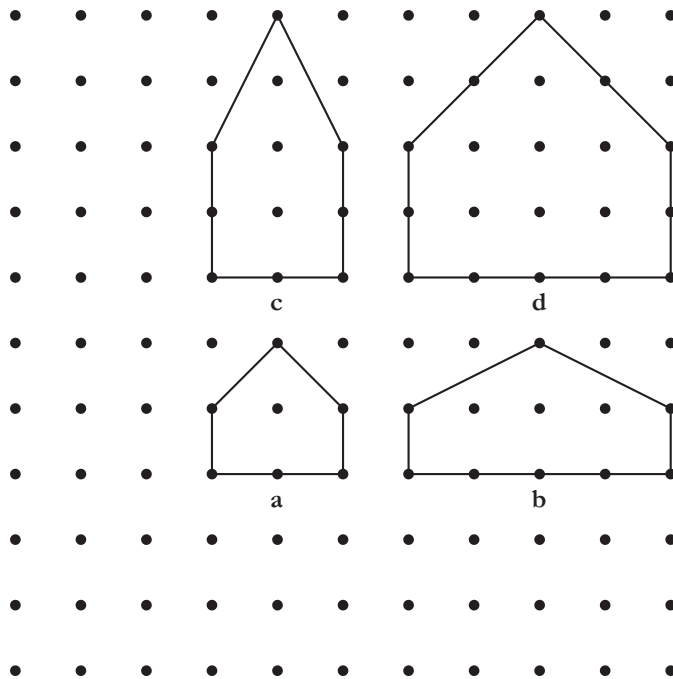
What is the missing number? Explain.

LAB 10.1

Scaling on the Geoboard

Name(s) _____

■ **Equipment:** 11×11 geoboard, dot paper



The figure shows four houses. House **a** is the original, and the others have been copied from it following very exact rules.

1. What is the rule that was used for each copy?

2. Among the three copies, two are distorted, and one is a scaled image of the original.
 - a. Which one is scaled?
 - b. How would you describe the distortions?

When two figures are scaled images of each other, they are said to be *similar*. Similar figures have *equal angles* and *proportional sides*. The sides of one figure can be obtained by multiplying the sides of the other by one number called the *scaling factor*.

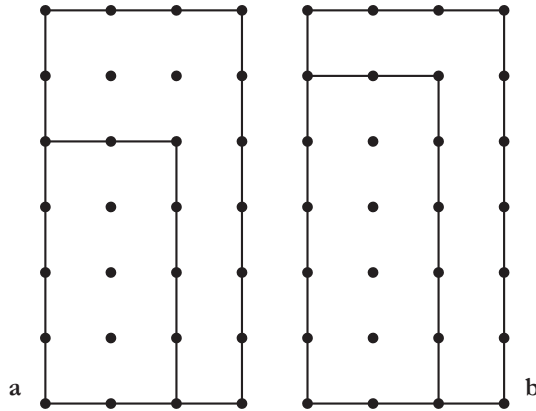
3. What is the scaling factor that relates the two similar houses in the figure above?

LAB 10.2

Similar Rectangles

Name(s) _____

■ **Equipment:** 11×11 geoboard



The figure above shows geoboard rectangles nested inside each other.

1. Explain why the two rectangles on the left are similar but the two rectangles on the right are not.

2. Draw the diagonal from the bottom left to the top right of the larger rectangle in **a**. Note that it goes through the top right vertex of the smaller rectangle. Repeat on **b**. How is it different?

3. The rectangles in **a** are part of a family of similar geoboard rectangles. If we include only rectangles with a vertex at the origin (the bottom left peg of the geoboard), the family includes ten rectangles (five horizontal ones and five vertical ones—two of the vertical ones are shown in the figure).
 - a. List the rectangles in that family by listing the coordinates of their top right vertex.

 - b. What is the slope of the diagonal through the origin for the vertical rectangles?

 - c. What is the slope of the diagonal for the horizontal rectangles?

LAB 10.2

Similar Rectangles (continued)

Name(s) _____

4. Including the one in Problem 3, there are ten families of similar rectangles (including squares) on the geoboard. Working with your neighbors, find them all, and list all the rectangles in each family. (Again, assume a vertex at the origin, and use the coordinates of the top right vertex for the list. Only consider families with more than one rectangle.)

5. Working with your neighbors, find every geoboard slope between 1 and 2. (Express the slopes both as fractions and as decimals.) Counting 1 and 2, there are seventeen different slopes.

Discussion

- A. Problem 2 is an example of using the *diagonal test for similar rectangles*. Explain.
- B. In Problems 3 and 4, how does symmetry facilitate the job of listing the rectangles?
- C. What are the advantages of fractional versus decimal notation in Problem 5?
- D. Explain how to use the answers to Problem 5 to create lists of geoboard slopes in the following ranges.
 - a. Between 0.5 and 1
 - b. Between -1 and -2
 - c. Between -0.5 and -1

Discounts

A discount card at a movie theater costs \$10. With that card, it only costs \$3 to attend a movie, instead of \$5. The card is valid for three months.

1. Use the same pair of axes for both of the graphs in this problem. Make a graph of the *total cost* (including the cost of the discount card if you got one) as a function of the *number of movies* you see:
 - a. if you have the discount card.
 - b. if you do not have the discount card.
2. What is the total cost of seeing n movies in three months
 - a. with the discount card?
 - b. without the discount card?
3.
 - a. Which of those two choices is a proportional relationship?
 - b. **Bonus:** Can you think of another situation involving money that is a proportional relationship?
4.
 - a. If you saw 12 movies in three months, how much would you save by buying the discount card?
 - b. If you saw only 2 movies in three months, how much would you save by *not* buying the discount card?
5. What is the break-even point; that is, how many movies would you have to see in order to spend exactly the same amount with and without the discount card?
6. How would your decision be affected if the cost of the discount card were raised to \$12?
7. **Challenge:** How would your decision be affected if the cost of the discount card were changed to \$ D ?

Guess My Function

Here is an example of a function: $y = 2x - 3$. For this function,

If $x = 0$, $y = -3$.

If $x = 1$, $y = -1$.

1. a. If $x = 2$, $y = ?$
- b. If $x = 1.5$, $y = ?$
- c. If $x = -1$, $y = ?$

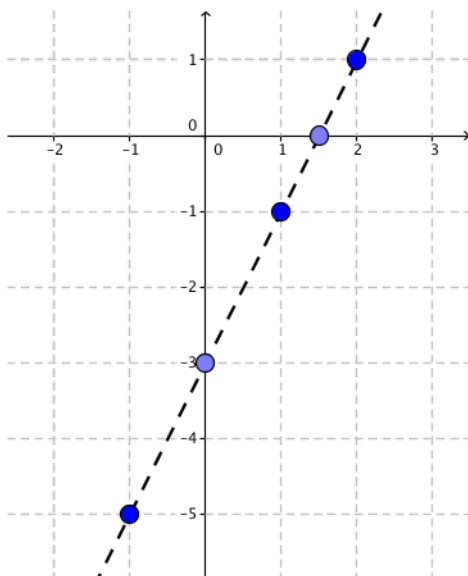
x is called the *independent variable*, or input. For this function, you can choose any number for x . y depends on x , so it is called the *dependent variable*, or output.

You can arrange the information about this function in a table:

x	y
-1	
0	-3
1	-1
1.5	
2	

2. Fill out the rest of the table.

You can also arrange the information in a graph:



3. Label the points with their coordinates.

Definition: A *function* is a rule that assigns to each input exactly one output.

On this page, we made a table and a graph from knowing the formula for the function.

On the next page, you will guess formulas for functions, knowing a table or a graph.

4. Guess a formula of the function for each table. (Hint: what can you do to x to get y ?)

a.

x	y
-2	0
-1	1
0	2
1	3
2	4

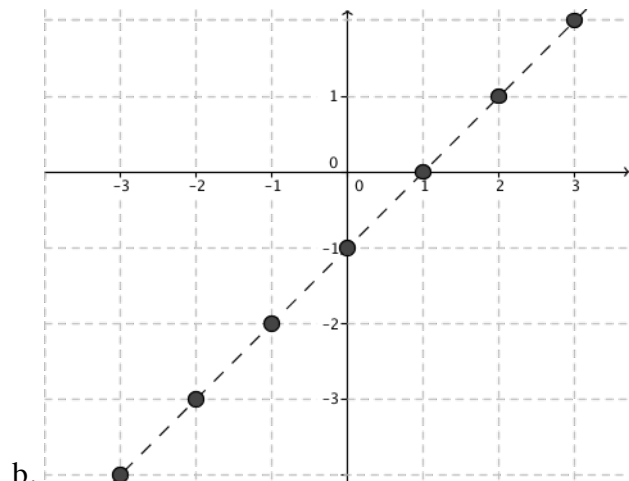
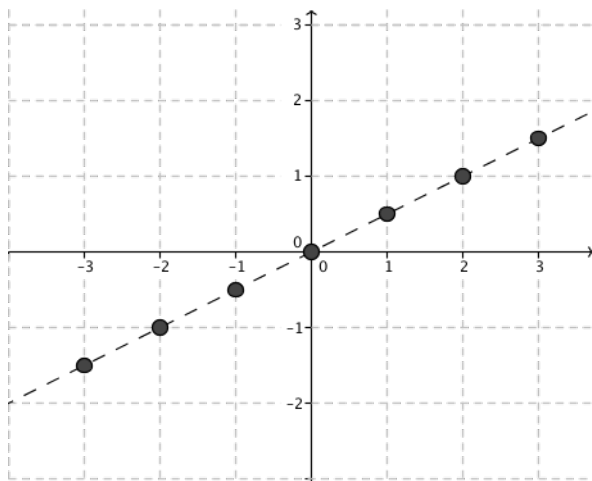
b.

x	y
-2	-6
-1	-3
0	0
1	3
2	6

c.

x	y
-2	-6
-1	-5
0	-4
1	-3
2	-2

5. Guess the formula of the function for each graph. (Hint: label the points with their coordinates.)



6. Which functions in #4 and #5 are proportional relationships?

7. Guess a formula of the function for each table. These are more challenging.

a.

x	y
-2	6
-1	5
0	4
1	3

b.

x	y
-2	-7
-1	-5
0	-3
1	-1

c.

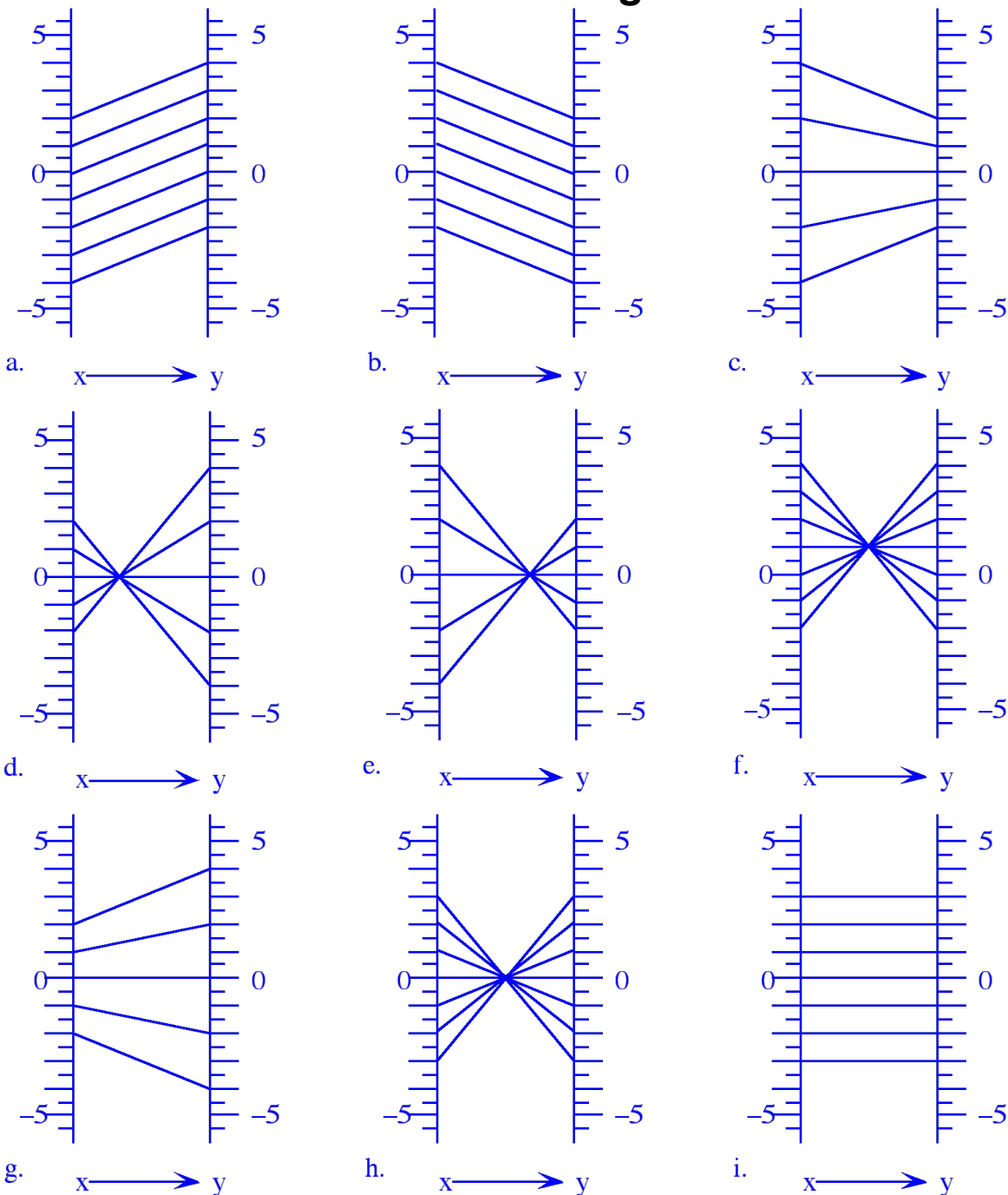
x	y
-2	3
-1	0
0	-1
1	0

Functions appear in all sorts of situations in math. For example, the input could be the side of a square, and the output its area. In that case, the formula would be $A=s^2$.

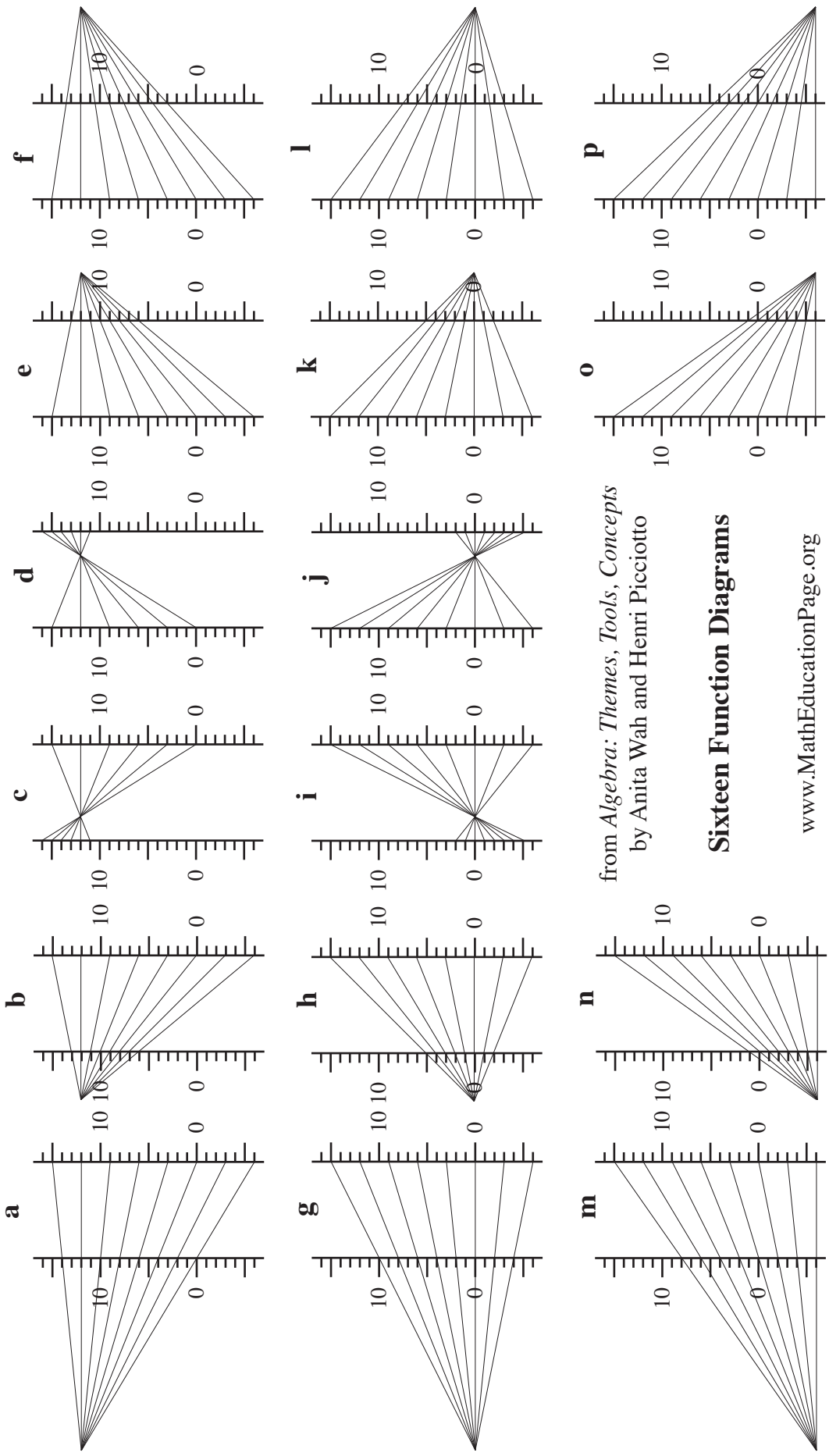
8. For each function, write a formula, and name the input and the output. Which are proportions?

- The perimeter of a square
- Half of a number
- The area of a circle

Nine Function Diagrams



1. Three diagrams represent functions of the form $y = x + b$. Which ones? What is b ?
2. Three diagrams represent functions of the form $y = x - b$. Which ones? What is b ?
3. Six diagrams represent functions of the form $y = mx$. Which ones? What is m ?
4. Six diagrams represent functions of the form $y = x/m$. Which ones? What is m ?
5. A function diagram has parallel in-out lines. Write as much as you can about the function.
6. If the in-out lines are parallel, in what case do they go up? Down? Straight across?
7. A diagram for $y = mx$ has in-out lines that move closer to each other. What can you say about m ?
8. A diagram for $y = mx$ has in-out lines that move apart from each other. What can you say about m ?
9. A function diagram for $y = mx$ has in-out lines that cross each other. What can you say about m ?
10. Two diagrams above represent functions of the form $y = b - x$. Which ones? What are the functions?

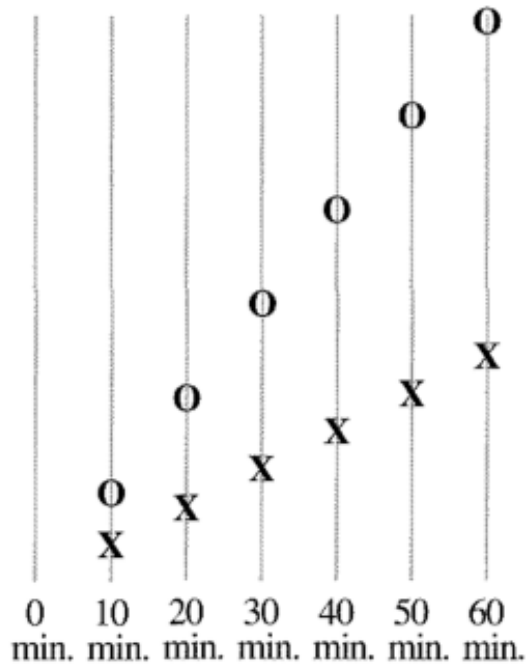


from *Algebra: Themes, Tools, Concepts*
by Anita Wah and Henri Picciotto

Sixteen Function Diagrams

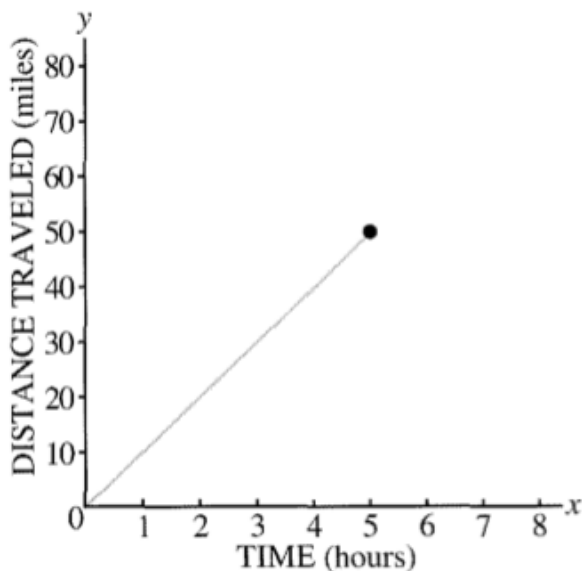
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Constant Speed



- Ophelia and Xavier are traveling along a road. If you could view the road from above and make a sketch of what you saw every ten minutes, your sketches might look something like the figure.
 - Which person (O or X) is traveling faster?
 - If the entire length of the road is six miles, can you figure out approximately how fast each person is traveling? Explain.

Bea is participating in a long-distance roller-skating race. Her speed is approximately 10 miles per hour. The graph below shows Bea's progress. It shows that after 5 hours she had traveled 50 miles.



- Bea's motion:
 - Copy the graph onto graph paper. Use a whole piece of graph paper. You will be adding more to this graph.
 - One of the points on the graph is (5,50). Mark and label three more points on the graph of Bea's progress.
- The distance traveled and the time elapsed are in a proportional relationship. Explain how we know this.
- In this lesson we are assuming everyone travels at a constant speed.
 - What might make it impossible to travel at a constant speed? Explain.
 - Why might it be a good idea to assume constant speed anyway?

Abe is walking along the same road. Assume he is moving at an approximately constant speed. The table shows how long it took for Abe to go certain distances.

Abe's Progress

Time (hours)	Distance (miles)
1	4
2	8

5. a. Copy and complete the table up to 20 miles.
b. Use the same axes you used for Bea. Plot and label the points from the table in part (a).
c. Connect the points with a straight line. Then find and label a point that is on the line but not in your table. Interpret the coordinates of the point in terms of this problem.

Amazingly, Gabe and Al are on the same road, and started at the same time. Gabe is riding a scooter, going 30 mph. Al is driving a van, going 50 mph.

6. Make tables like the one you made for Abe showing Gabe's progress on his scooter and Al's progress in the van. Make graphs of their progress on the same axes you used to show Abe's and Bea's progress. Label the four different lines.
7. Use your graphs to help you answer these questions. If Bea and Abe start out at the same time,
a. how far apart will they be after one hour?
b. how far apart will they be after two hours?
8. Look for a pattern.
a. How far apart will Abe and Bea be after H hours? Explain.
b. Is this a proportional relationship? Explain.
9. Mrs. Gral was traveling at a constant speed. She started on the same road at the same time as all the others, and was two miles ahead of Abe after one hour.
a. Add a graph of Mrs. Gral's progress to your axes.
b. How far ahead was Mrs. Gral after two hours?
c. Was she ahead of, or behind Bea? After three hours, how far ahead or behind?
d. How fast was Mrs. Gral going? What mode of travel do you think she was using?
10. **Summary:**
a. How does the mode of travel affect the steepness of the line? Explain.
b. What is the meaning of points on two of the graphs that have the same x -coordinate but different y -coordinates?
c. What is the meaning of the vertical distance between two lines for a given value of x ?

In the Lab

A Mystery Liquid

Reg, Bea and Gabe were doing an experiment in science class. They had an unknown liquid whose volume they measured in a graduated cylinder. A graduated cylinder is a tall, narrow container that is used for accurately measuring liquid volume. They used one that weighed 50 grams, and measured volume in milliliters. They used a balance to find the mass of the liquid to the nearest gram.

Reg's data

volume	mass
10 ml	16 g
20 ml	32 g
50 ml	80 g
80 ml	128 g

- Plot Reg's data, with *volume* on the horizontal axis and *mass* on the vertical axis.
 - Does it make sense to connect the points on your graph? Explain.
- Find an equation relating mass to volume.
- Estimate the mass of
 - 60 ml of liquid
 - 1 ml of liquid
- If you add 30 ml to the volume, how much are you adding to the mass? See if you get the same answer in two different cases.
- If you double the volume, do you double the mass?

Bea's data

mass	volume
16 g	10 ml
32 g	20 ml
48 g	30 ml
64 g	40 ml

- Plot Bea's data on the same axes, using a different symbol or color for those points. Be careful! Because the mass depends on the volume, make sure the volume is on the horizontal axis, and the mass is on the vertical axis, again.
 - Connect the points on your graph with a line and write an equation for the line.
 - Estimate the volume of:
 - 100 g of liquid
 - 1 g of liquid
 - Compare Bea's graph with Reg's graph. Explain the similarities and differences.
 - If you add 10 ml to the volume, how much are you adding to the mass? See if you get the same answer in three different cases. Is the answer consistent with what you found in Reg's data?
- Definition:** *Density* equals mass per unit of volume. This means that to find the density of the mystery liquid, you would find the mass of 1 ml of the liquid.
- Find the density of the mystery liquid, using three different pairs of mass / volume values from Reg's and Bea's data. Do all your answers agree? Explain.
 - Is the relationship between volume and mass a proportional relationship? Explain.

The Mystery Grows

Gabe's data:

volume	mass
10 ml	66 g
20 ml	82 g
40 ml	114 g
60 ml	146 g

13. a. On the same axes, plot Gabe's data.
b. In Gabe's table, if you double the volume, does the mass double? Check this in two cases.
14. If you add 20 ml, how much mass are you adding? Is this consistent with what you learned from Reg's and Bea's data?
15. a. *According to Gabe's graph*, what is the mass of 0 ml of the liquid? Does this make sense?
b. What might be the real meaning of the y-intercept on Gabe's graph? Did Gabe make a mistake? Explain.
16. Divide mass by volume for three different pairs of values from Gabe's data. Do all your answers agree? Will that method work to find the density of the liquid?.
17. Are the numbers in Gabe's table examples of a proportional relationship? Explain.
18. Write an equation that expresses mass as a function of volume for Gabe's data.
19. Compare Gabe's graph to Reg's and Bea's. How are they the same, and how are they different?

20. There are number patterns and graph patterns in all the data in this lesson.
 - a. What pattern is there in all of Reg's, Bea's, and Gabe's data?
 - b. What patterns are only true of Reg's and Bea's data?

Other Substances

21. The graph shows the relationship between mass in grams, and volume in milliliters for some familiar substances. The substances are aluminum, cork, gold, ice, iron and oak. Which substance do you think is represented by each line? Explain.

