Chapter 2 Computing with Signed Numbers

This chapter is an introduction to integer arithmetic.

New Words and Concepts

The usual approach to work with signed numbers (teaching of rules, sometimes combined with number line techniques) is often frustrating, because so many students seem to forget the rules, even though they are given endless practice. On the other hand, the students who "get it," quickly get bored with the drills.

The Algebra Lab features a different approach. Instead of being taught signed number arithmetic, the students are presented with a model for the four operations. The model is easy to learn, because it is a natural extension of concrete models most students have already mastered for these operations. In fact, once learned, the model is almost impossible to forget, unlike the rules and number line techniques, which are confusing and easy to mangle. Moreover, the work with the blocks keeps more students motivated, at all levels of proficiency.

The commutative law is not mentioned by name, but students are asked to observe what happens when switching the order of the terms for each operation.

After working with the blocks for a while, you will find that many students discover the rules for themselves, or learn them from each other. Knowledge that is earned the hard way by one's own thinking and interaction with peers is much more securely anchored than techniques that have been learned by rote.

The geometric concepts of **perimeter**, **area**, **volume**, and **surface area** appear for the first time in this chapter.

Teaching Tips

In grades K-4, students should have been introduced to whole number, fraction, and decimal arithmetic with manipulatives such as Base Ten Blocks. Such background is certainly helpful in understanding the manipulative approach to the arithmetic of signed numbers.

If your students already know signed number arithmetic, there is little point in working through this chapter, except for Lesson 4 and the Explorations. An interesting approach with such students is to ask them to figure out a way to use the Lab Gear to explain the rules of signed number arithmetic. If they cannot figure it out, show them the models, but do not assign too many drill problems.

In general, students should be able to perform most arithmetic calculations in this program mentally. However, a student whose arithmetic is shaky should not be prevented from learning algebra. Using a calculator should allow such a student to temporarily get around this handicap. Of course, all students should also be allowed to use calculators if they need them on lengthy, multi-digit computations.

Do not make the mistake of skipping Explorations 2 and 3! The geometric content is an enhancement to an algebra course, and offers opportunities to apply algebraic ideas.

Lesson Notes

- **Lesson 1,** Addition, page 16: This lesson is based on the cancelling concept from Chapter 1, Lesson 7.
- **Lesson 2,** Subtraction, page 18: This lesson is based on the "uncancelling" concept from Chapter 1, Lesson 7.
- **Lesson 3,** Multiplication, page 21: The "rectangle" interpretation of multiplication is introduced here. This is a crucial conceptual tool, which will be developed further (in conjunction with the Lab Gear corner piece) starting in Chapter 3.
- **Lesson 4,** Squaring Numbers, page 23: Most students will not learn the concepts on this page by just reading it. It is important that there be some verbal discussion of these ideas.
- **Lesson 5,** Generalizing, page 24: Later in the year, if your students have trouble with signed number arithmetic, refrain from reminding them of the rules. This would only perpetuate their dependence on you. Instead, allow them to use the blocks to refresh their memories, or to look up what they wrote at the end of this chapter. In most cases, this will do the trick.
- **Lesson 6,** Division, page 25: The key to this lesson is a solid understanding of multiplication, and of the relationship between the two operations. See Chapters 3-7 for Lab Gear division techniques on algebraic expressions that involve variables.

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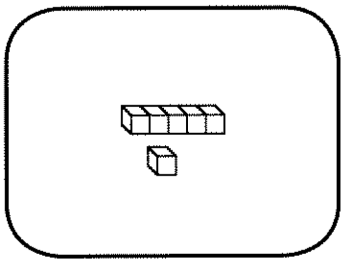
Addition

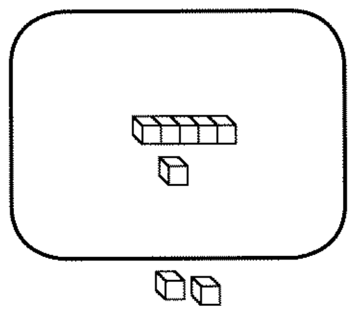
To model addition with the Lab Gear, *put on* the first number, *put on* the second number, *cancel* what you can, and *count*.

Look at this example, –6 + 2.

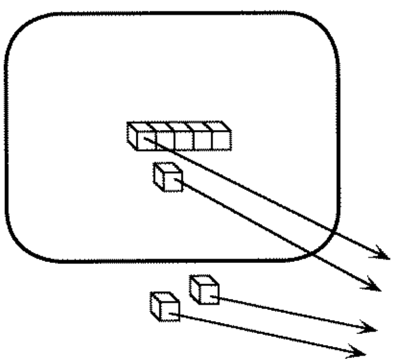
- *Put on* the first number. To show –6, put 6 in the minus area.



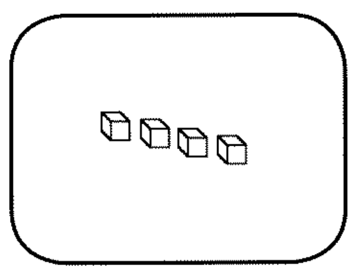
- *Put on* the second number. To show 2, put 2 outside the minus area.



- *Cancel* 2 and –2.



- *Count* to find the answer. –6 + 2 = –4



1. Does the order that you put on the numbers matter? Try it by modeling 2 + –6, and compare the result.

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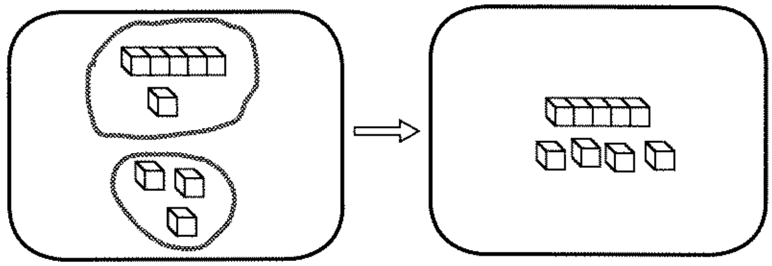
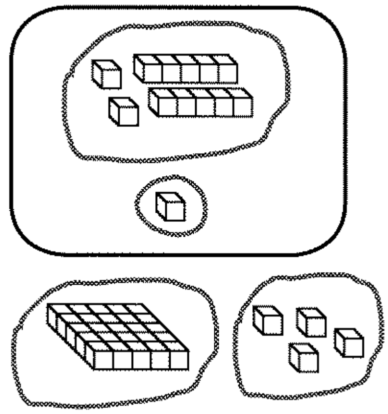
Sometimes, there is nothing to cancel.

2. Write the addition shown by this figure.



If you are adding several numbers, just put them all on the mat, cancel, and count.

3. Write the addition shown by this figure.



Use the Lab Gear to model these additions. Sketch the process in four steps (put on first number / put on second number / cancel / final solution). The sketches should include the minus area. You can sketch the blocks in two dimensions.

4. –5 + 2
5. –15 + (–3)
6. –8 + 9

Use the Lab Gear to find these sums.

7. 10 + –4
8. –25 + 5
9. –12 + –6
10. 10 + (–8) + 3
11. –25 + 11 + (–4)
12. –15 + 20 + (–7) + 2

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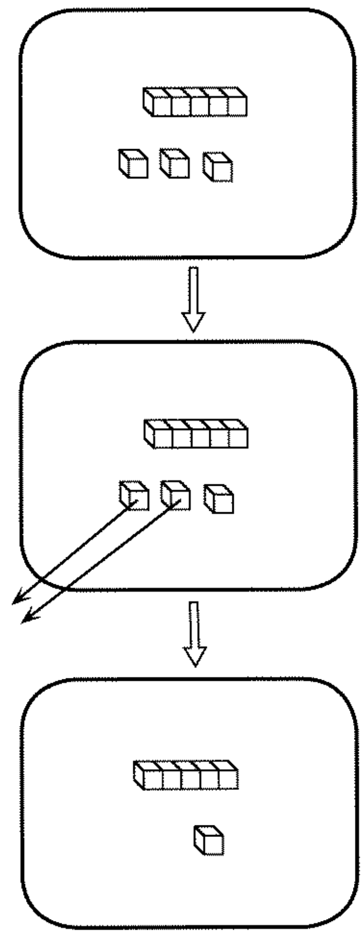
Subtraction

To model subtraction with the Lab Gear, *put on* the first number, *take off* the second number, and *count.*

This example shows that –8 – (–2) = –6.

- Put on –8.
- Take off –2.
- Count to find the answer. –6



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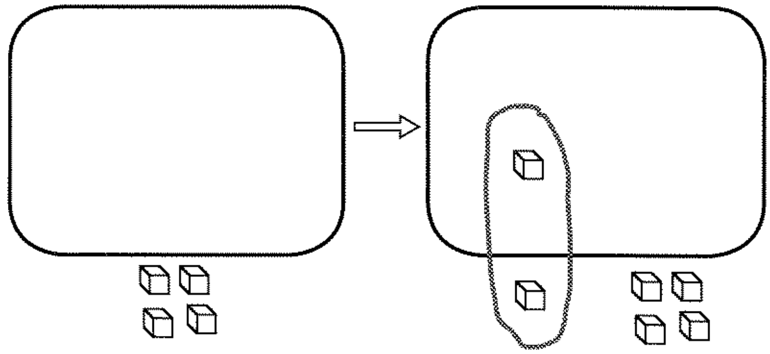
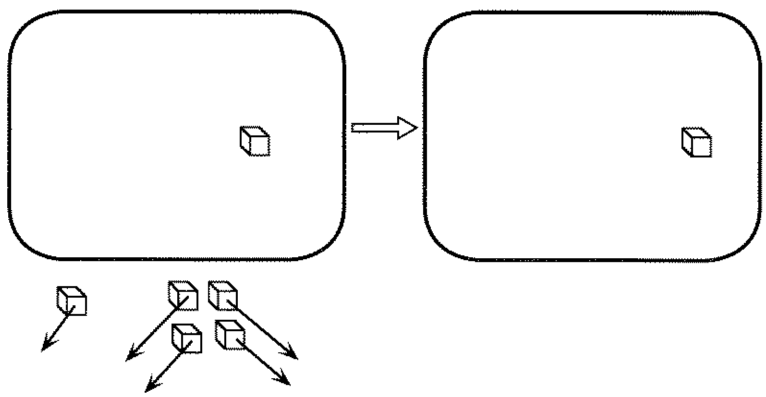
Sometimes, there are not enough blocks to take off. For example, 4 – 5. The solution is to show 4 differently by *adding zero*.



As long as you add the same amount inside the minus area and outside it, you have not changed the quantity on the workmat. If you cancelled the blocks you put on, you would be back to the original amount. (Adding zero could be called "uncancelling.")

Notice that now it is possible to take off 5, and you can see that the answer is –1.



Perhaps you knew what the answer was going to be, and you are asking yourself, why do so much work to get it? Keep in mind that the point of working with the Lab Gear is not just to get the answer, but to understand why it turns out the way it does. Work these easy problems carefully; you will soon be doing similar ones with variables. You will then see how working with variables is just an extension of the work you're doing now with integers.

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Use the Lab Gear to model these problems.

1. $-7 - (-3)$
2. $-3 - (-7)$
3. In problem 2, the order of the numbers was reversed. How did that affect the answer?

The adding zero trick should help you do some of these problems. For problems 4-7, sketch the process in four steps (put on / add zero (if you need to) / take off / final solution).

4. $-2 - 8$
5. $-4 - (-2)$
6. $-8 - 3$
7. $7 - (-6)$
8. $-5 - 9$
9. $11 + (-6)$
10. $-1 + 10$
11. $20 - 25$
12. $13 - 3$
13. $-17 + 3$
14. $-2 - 21$
15. $-15 - 5$
16. $-5 - (-15)$
17. $6 - 17$
18. $18 - 9$

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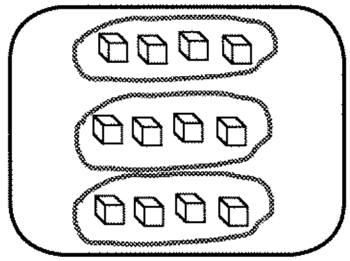
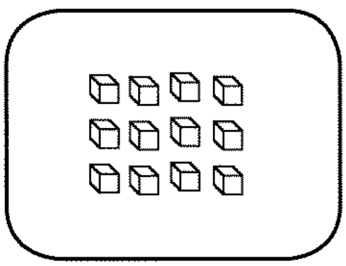
Multiplication

To model multiplication with the Lab Gear, *always start with an empty workmat.* For the first example, consider the multiplication 3 (–4).

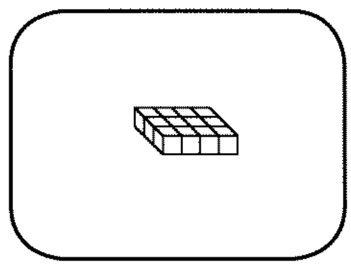
- Look at the first number. If the first number is positive, *put on* the second number the number of times indicated by the first number. Here you put on three sets of –4.



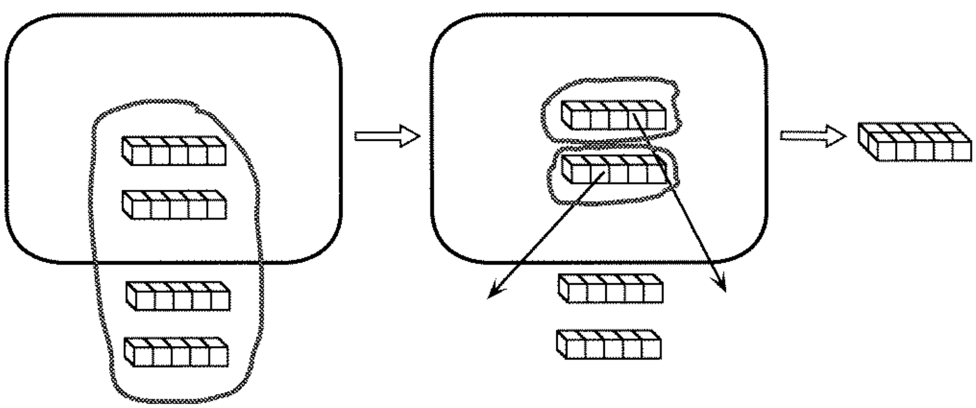
- *Count* to get the final result. –12



- *Make a rectangle* to show multiplication. 3 (–4) = –12



For the second example, consider the multiplication –2 (–5).



- If the first number is negative, *take off* the second number the number of times indicated by the first number. However, since you're starting with an empty mat, there is nothing to take off, so you must use the adding zero trick. In this case, add 10 and –10, then *take off* two sets of –5.
- *Count* to get the final result.
- *Make a rectangle.* –2 (–5) = 10.

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Look at the examples on page 21. Use the Lab Gear in the same way to model problems 1 and 2. Notice that the order of the numbers has been reversed.

1. –4 (3)
2. –5 (–2)
3. Tell how changing the order affects the answer.

Notice that at the end of a multiplication problem, you can always arrange the blocks into a rectangle.

Use the Lab Gear to model these multiplications. For problems 4-6, sketch the process in two or three steps. (Put on / make a rectangle; or, add zero / take off / make a rectangle.)

4. 2 (–4)
5. –3 (2)
6. –4 (–5)
7. –6 (–2)
8. –10 (2)
9. 7 (–3)

Compute:

10. –9 + 6
11. 2 – (–4)
12. –3 – 2
13. –3 (–2)
14. Copy and complete this sentence: Problem 12 is a subtraction, while problem 13 is a __________.

Multiplying by –1

Use the Lab Gear to model these multiplications.

15. –1 (–2)
16. –1 (3)
17. 4 (–1)
18. –5 (–1)
19. Explain what happens when a number is multiplied by –1.
20. What happens *to the blocks* when you multiply a number by –1? For each example above, sketch the blocks that represent the number *before* and *after* multiplying by –1. Describe what happens in each case.

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Squaring Numbers

A special case of multiplication is the multiplication of a number by itself. Use the Lab Gear to model these multiplications. At the end of each problem, check whether the answer is positive or negative, and try to arrange the blocks into a square.

1. $(-3)(-3)$
2. $5 \cdot 5$
3. $(-2)(-2)$
4. $(-6)(-6)$

Multiplying a number by itself is called *squaring* the number. Instead of writing $5 \cdot 5$, you can write 5^2, which is read *five squared.* Instead of writing $(-5)(-5)$, you can write $(-5)^2$. This is read *negative five, squared*, or perhaps more clearly, *the square of negative five.* The parentheses tell us that we are squaring –5. The answer is 25.

Be careful! If you write -5^2, without the parentheses, that will be read as *the opposite of five squared,* in other words, the opposite of 25, or –25. In conclusion, $(-5)^2$ and -5^2 are not the same! In fact, they are opposite. If you want to indicate squaring –5, you must use parentheses.

Write how to read each of the following expressions.

5. 2^2
6. $(-4)^2$
7. -3^2
8. $(3x)^2$
9. $(-3x)^2$
10. $-3x^2$
11. In problems 8, 9, and 10, which expressions are equal to each other? Explain your answer. Hint: Work out the numerical value for each one if $x = 2$, and then if $x = -2$.

When writing 5^2, the 2 is called the *exponent*, and the 5 is called the *base*. When writing $6x^2$, the 6 is called the *coefficient*.

12. In the expression 7^2, what do you call the 2? the 7?
13. In the expression $3xy$, what do you call the 3?

Exploration 1 Positive or Negative?

Look at these expressions. For each one, indicate with 0, P, and/or N, if the value of the quantity is zero, positive, or negative. If it is impossible to tell, write ?.

1. 2^2
2. $(-4)^2$
3. -3^2
4. $(3x)^2$
5. $(-3x)^2$
6. $-3x^2$

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Generalizing

While using the Lab Gear to do integer arithmetic, you may have discovered some rules, tricks, or patterns that will help you deal with minus signs when doing arithmetic without the blocks. Write down any such rules or patterns.

1. Addition rules, tricks, or patterns
2. Subtraction rules, tricks, or patterns
3. Multiplication rules, tricks, or patterns
4. Squaring rules, tricks, or patterns

You have been working with integers. However, the rules for integer arithmetic still apply when working with decimals and fractions. For example, the square of a negative fraction, just like the square of a negative integer, is always positive. Try to do the following computations without pencil or calculator:

5. $-5.2 + 11.36$
6. $-2.2 - 0.06$
7. $-3.07\ (1000)$
8. $-\frac{2}{3} + \frac{1}{6}$
9. $-\frac{2}{3} - \frac{1}{6}$
10. $-\frac{2}{3} \cdot \frac{1}{6}$

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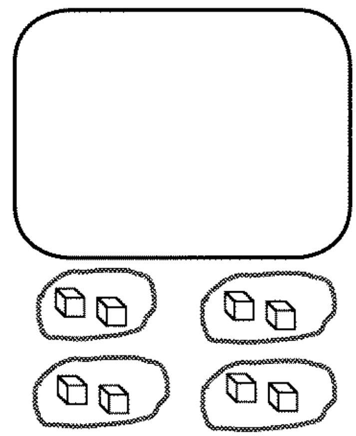
Division

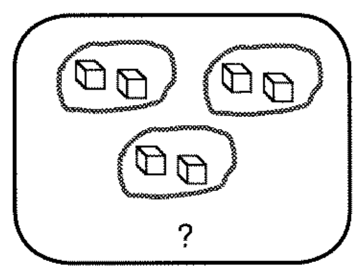
You may be wondering how division is performed with the Lab Gear. Later, you will learn to use the Lab Gear to divide some expressions that involve variables. For integers, the best method is to use the fact that division is the inverse operation of multiplication.

For example, to divide 8 by 2, you could ask, "What times 2 equals 8?" Remember that for multiplication, you start with an empty mat. How many sets of 2 should you *put on* to get 8? It is easy to see that the answer is 4.

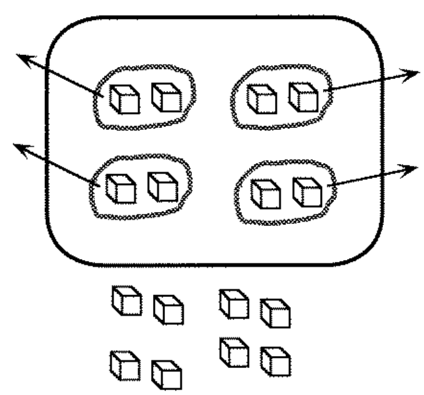


1. Consider the division –8 ÷ –2. What is the multiplication question to ask? How would you solve it on the workmat? Write the answer.

Now consider dividing 8 by –2. Ask yourself, "What times –2 equals 8?" Start with an empty workmat. How many sets of (–2) should you *put on* to get 8? Clearly that question has no answer, since no matter how many times –2 is *put on* you will only get negative numbers.



Try another approach. How many sets of (–2) should you *take off* to get 8? Start with an empty workmat and add zero in the form of 8 and –8. Now try *taking off* four sets of –2 to leave 8. This works. So the answer to "What times –2 equals 8?" is –4. So, 8 ÷ –2 = –4.



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2. Consider the division $-8 \div 2$. What is the multiplication question to ask? How would you solve it on the workmat? Write the answer.

3. Using the method shown in the examples, try to figure out rules for division. How do negative numbers in the numerator and/or denominator affect the result? Give examples.

4. Compare these rules to the ones for multiplication.

5. What happens when a number is divided by –1?

6. Which of these divisions are equal?

   a. $\frac{8}{-2}$

   b. $\frac{-8}{2}$

   c. $\frac{-8}{-2}$

   d. $\frac{8}{2}$

   e. $\frac{-8}{2}$

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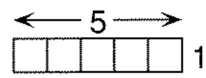
Exploration 2 Area and Perimeter

The **perimeter** of a two-dimensional figure is the number of units of length around the figure. The unit of length we will use is the centimeter (cm).

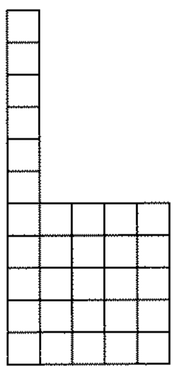
The **area** of a figure is the number of square units it would take to cover it. The unit of area we will use is the square centimeter (cm^2). When we discuss the perimeter and area of the Lab Gear blocks, we will be thinking of the tops of the blocks, which are flat, two-dimensional figures.

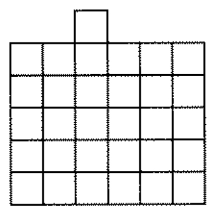
For example, if you look at the 5-block from above, you would see this rectangle. Its area is 5 cm^2, and its perimeter is 12 cm.

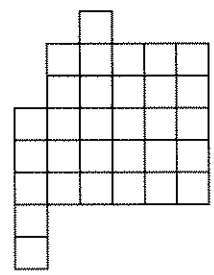


Area: 5 cm^2
Perimeter:
$5 + 1 + 5 + 1 = 12$ cm

Find and write the area and perimeter of these figures, which are the top faces of groups of yellow blocks.

1. 

2. 

3. 

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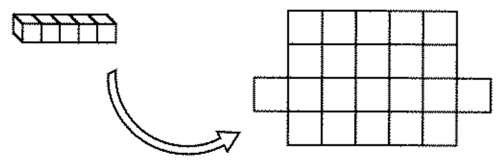
Exploration 3 Volume and Surface Area

The **volume** of a block is the number of cubic units it would take to build it. The unit of volume we will use is the cubic centimeter (cm^3).

For example, the volume of the 5-block is 5 cm^3.

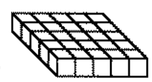
The **surface area** of a block is the number of square units it would take to cover all its faces. To figure out the surface area, it helps to think of a paper jacket that would cover the whole block. The area of such a jacket is the surface area of the block.

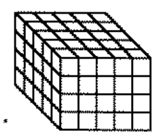
For example, the surface area of the 5-block is 22 cm^2.

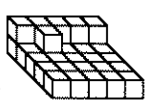


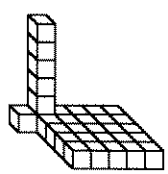
Find and write the volume and surface area of each of these buildings. Notice that in problems about surface area or volume, upstairs blocks do not mean minus.

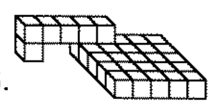
1. 

2. 

3. 

4. 

5. 

6. 

(For the remaining problems in this binder, the units will always be assumed to be cm, cm^2, or cm^3 for length, area, and volume, respectively.)

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