

Iterating Linear Functions

These questions are about iterating $f(x) = mx + b$. Seed: x_1 . Orbit: x_1, x_2, x_3, \dots

For each question, give examples and a symbolic answer.

1. What values of m and b yield
 - a. an arithmetic sequence?
 - b. a geometric sequence?
2. What happens if
 - a. $m = 0$?
 - b. $m = -1$?

A *fixed point* f is a value of x such that its orbit is f, f, f, \dots

3. When iterating a linear function,
 - a. if there is a fixed point, find a formula for it
 - b. when does it not exist?
4. For what values of m and b is the fixed point
 - a. attracting? (all orbits have the fixed point as a limit)
 - b. repelling? (all orbits other than the fixed point's move away from it, towards infinity)
5. Find the differences (d_1, d_2, d_3, \dots) between successive iterates and the fixed point. What sort of sequence do you get? (arithmetic? if so, what is the common difference? geometric? if so, what is the common ratio? neither? Try this at least once with $m < 1$, and once with $m > 1$.)
6. Find an explicit formula for d_n .
7. Find an explicit formula for x_n .

Selected Answers

$$3. f = \frac{b}{1-m}$$

$$\begin{aligned} 5. d_n &= f - x_n \\ d_{n+1} &= f - x_{n+1} = f - (mx_n + b) \\ &= mf + b - mx_n - b && (\leftarrow \text{since } f \text{ is fixed}) \\ &= m(f - x_n) = md_n && (\text{so the distances make a geometric series}) \end{aligned}$$

$$\begin{aligned} 6. d_0 &= f - x_0 \\ d_n &= d_0 m^n = (f - x_0) m^n \end{aligned}$$

$$\begin{aligned} 7. x_n &= f - d_n \\ &= f - (f - x_0) m^n \end{aligned}$$