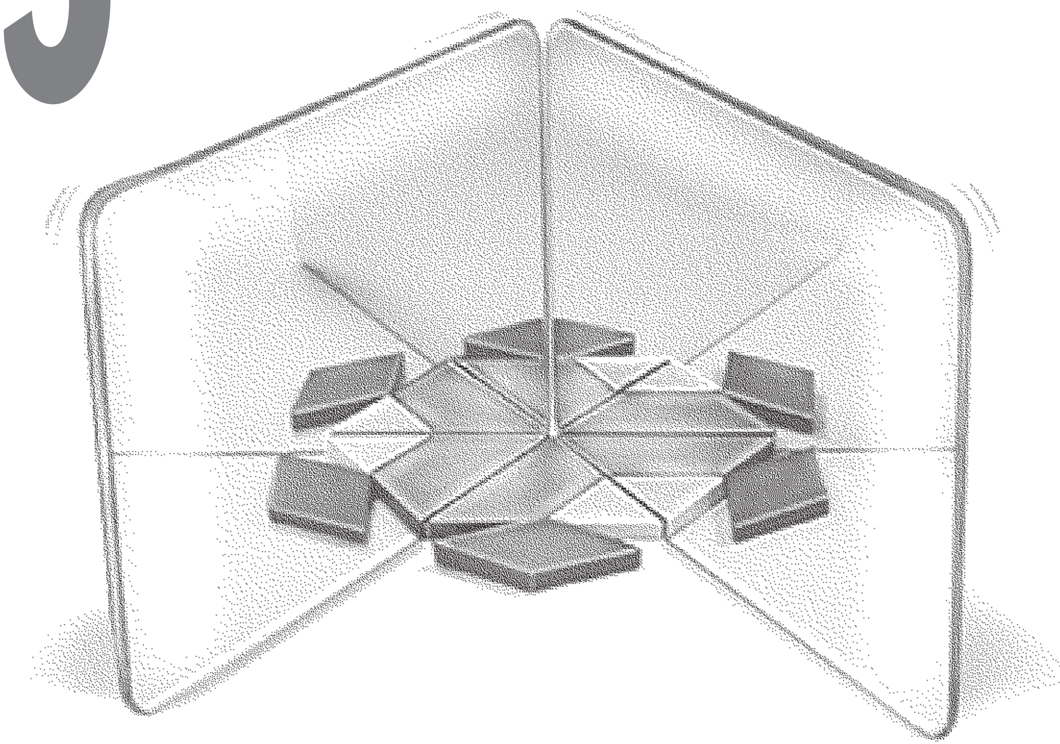
5 Symmetry



Many of the activities in this section present opportunities for students to express themselves mathematically and artistically—you should be able to create many bulletin board displays from the work they do in various labs. Perhaps you should plan to put together a final exhibit based on the best work from throughout this section. (See the teacher notes to Lab 5.5 for ideas on symmetry exhibits.)

Note that the first lab is a prerequisite for all the other ones.

An excellent resource on the mathematics of symmetry and on crystallographer's notation for symmetric patterns is *Handbook of Regular Patterns: An Introduction to Symmetry in Two Dimensions* by Peter S. Stevens. In addition to a very clear exposition of the ideas, the book includes thousands of symmetric images from just about every culture on the planet. Martin Gardner's *The Ambidextrous Universe* is another good reference on symmetry. It opens with the question, "Why do mirrors reverse left and right, but not up and down?"

See page 191 for teacher notes to this section.

LAB 5.1
Introduction to Symmetry

Name(s) ______________________

1. These letters are *line symmetric*:

A D H I M O T U V W X Y

These letters are not:

F G J L N P R S Z

   a. Explain the difference.

   b. Draw the line or lines of symmetry on each letter in the first group. (In some cases, there are two lines.)

2. Show how each of the five capital letters not shown in Problem 1 can be written two ways: line symmetric, or not.

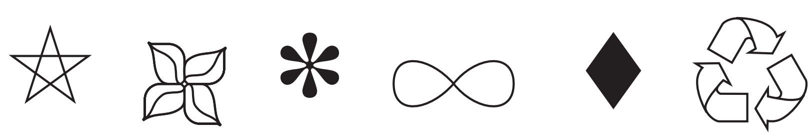
3. Organize the lowercase letters in two lists below: line symmetric, and not.

   Line symmetric:

   Not line symmetric:

4. Below are some figures that have rotation symmetry and some figures that do not. Explain the difference.

Have rotation symmetry



Do not have rotation symmetry

5. Organize the capital letters in two lists below: those with rotation symmetry and those without.

Rotation symmetry:

No rotation symmetry:

6. Organize lowercase letters in two lists below: those with rotation symmetry and those without.

Rotation symmetry:

No rotation symmetry:

7. Artist Scott Kim finds ingenious ways to adapt letters so that he can design symmetric words. Describe the symmetry of the design below.

From *Inversions,* by Scott Kim.

8. On a separate sheet of paper, write symmetric words. In this case, symmetry can be interpreted in several ways, using capitals or lowercase letters. The following list gives possible interpretations.

a. Words that have a horizontal line of symmetry, such as COB.

b. Words that have a vertical line of symmetry, such as TOT. (To make this easier, you may write words vertically, with each letter beneath the previous one.)

c. Pairs of words that are mirror images of each other, such as box and pox (using a horizontal mirror).

d. Words that can be read upside down, such as SOS.

e. Pairs of words where one word is the other word upside down, such as MOM and WOW.

f. Palindromes, such as RADAR or RACE CAR (ignoring the space).

Discussion

A. Line symmetry is also called *mirror* symmetry, *reflection* symmetry, *bilateral* symmetry, and *flip* symmetry. Explain why each of these words is appropriate.

B. What happens if a line-symmetric figure is folded along the line of symmetry?

C. For the figures in Problem 4, draw the lines of symmetry across the figures that are line symmetric, and draw the center of symmetry in those that have rotation symmetry. Note that some of the figures have both kinds of symmetry.

D. The figures in Problem 4 with rotation symmetry include examples of 2-fold, 3-fold, 4-fold, 5-fold, and 6-fold rotation symmetry. Classify each figure according to the specific type of rotation symmetry it has. Explain what it means for a figure to have "*n*-fold" rotation symmetry.

E. If a figure looks unchanged when you rotate it around a point by 180°, it has a special kind of rotation symmetry that is also called *half-turn* symmetry, *central* symmetry, and *point* symmetry. Explain why each of these words is appropriate.

F. What is your definition of the word *symmetry*? Compare it with other students' definitions and the dictionary definition.

G. Which letters have both line and rotation symmetry? What else do they have in common?

H. Some letters are ambiguous and can be written either with or without line symmetry. Others can be written either with or without rotation symmetry.

- Draw and explain examples of these ambiguities.
- Explore how different fonts accentuate or detract from the symmetry of individual letters.

LAB 5.2

Name(s) ____________________

Triangle and Quadrilateral Symmetry

■ **Equipment:** Dot or graph paper, isometric dot or graph paper

Eight types of triangles

Equilateral (EQ)
Right isosceles (RI)
Acute scalene (AS)
Half-equilateral (HE)
Acute isosceles (AI)
Obtuse isosceles (OI)
Right scalene (RS)
Obtuse scalene (OS)

Eight types of quadrilaterals

Square (SQ): a regular quadrilateral
Rhombus (RH): all sides equal
Rectangle (RE): all angles 90°
Parallelogram (PA): two pairs of parallel sides
Kite (KI): two distinct pairs of consecutive equal sides, but not all sides equal
Isosceles trapezoid (IT): exactly one pair of parallel sides, one pair of opposite equal sides
General trapezoid (GT): exactly one pair of parallel sides
General quadrilateral (GQ): not one of the above

1. Find two examples of each kind of figure listed above, triangles and quadrilaterals, on dot or graph paper. Label them.
2. On each figure, draw the lines of symmetry if there are any.
3. On each figure, mark the center of symmetry if there is any rotation symmetry.
4. Fill out the table below by entering the types of triangles and quadrilaterals in the appropriate spaces (some spaces may be empty; others may have more than one entry).

	Rotation symmetry			
Line symmetry	**None**	**2-fold**	**3-fold**	**4-fold**
No lines				
One line				
Two lines				
Three lines				
Four lines				

Discussion

A. Is there a relationship between the numbers of lines of symmetry and the nature of the rotation symmetry? Explain.

B. For each empty cell in the chart in Problem 4, explain why it is empty. Comment on whether any figures exist with that kind of symmetry (not triangles or quadrilaterals).

C. In some cells, there are two or more figures. Do those figures have anything in common besides symmetry? Discuss.

D. You may have heard that "a square is a rectangle, but a rectangle is not necessarily a square." Does the classification of the quadrilaterals by symmetry throw any light on this statement? Can you use the classification by symmetry to find more statements of this type?

LAB 5.3
One Mirror

Name(s) ____________________

■ **Equipment:** Mirror, template

Using your template on a piece of unlined paper, draw a triangle. Then place the mirror on it in such a way as to make a triangle or quadrilateral. For example, starting with an equilateral triangle *ABC* it's possible to make a different equilateral triangle, a kite, or a rhombus.

Check off the figures you make in this manner in the table below, and draw the mirror line on the triangle as a record of how you did it. Also indicate which side you are looking from with an arrow. On the table, mark impossible figures with an X.

Figures made	**By using the mirror on**							
Triangles	**EQ**	**AI**	**RI**	**OI**	**AS**	**RS**	**HE**	**OS**
Equilateral (EQ)	✓							
Acute isosceles (AI)	×							
Right isosceles (RI)								
Obtuse isosceles (OI)								
Acute scalene (AS)								
Right scalene (RS)								
Half-equilateral (HE)								
Obtuse scalene (OS)								
Quadrilaterals	**EQ**	**AI**	**RI**	**OI**	**AS**	**RS**	**HE**	**OS**
General								
Kite	✓							
General trapezoid								
Isosceles trapezoid								
Parallelogram								
Rhombus	✓							
Rectangle								
Square								

Name(s) ______________________

Discussion

A. What polygons besides triangles and quadrilaterals can be made with a triangle and a mirror?

B. Which figures on the table cannot be made with a triangle and a mirror? Why?

C. What do all the figures that can be made have in common?

D. Which line-symmetric figures cannot be made? Why?

E. Which figures can be made from any triangle?

F. What is a good strategy for finding a way to create a given shape with a triangle and mirror?

G. How must the mirror be placed on the original triangle so the resulting figure is a triangle? Why?

H. It is never possible to make an acute isosceles triangle from an obtuse isosceles triangle. However, it is sometimes possible to make an obtuse isosceles triangle from an acute one. Why?

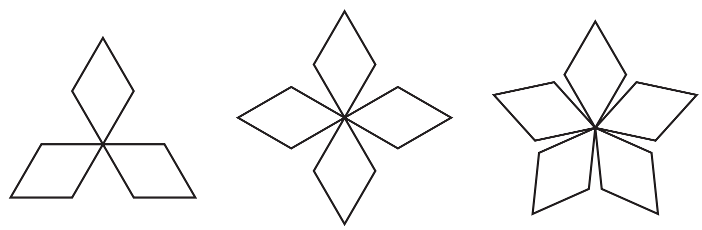
LAB 5.4
Two Mirrors

Name(s) ______________________

■ **Equipment:** A pair of mirrors hinged with adhesive tape, pattern blocks, template

In this lab, you will investigate what happens to reflections of pattern blocks as you change the angle between the mirrors.

1. Using your mirrors and one blue pattern block, make the figures below. To record your work, draw the mirrors' position on the figures.



2. For each example in Problem 1, what is the angle between the mirrors? Explain.

3. What is the next figure in the sequence? Make it with a block and mirrors, and draw it in the space at right. Is there another figure after this one?

4. Using a tan pattern block, set up the mirrors so you can see *exactly* 2, 3, 4, 5, and so on copies of it, including the original block (no matter where you look from). For each setup, what is the angle between the mirrors?

5. Calculate the following angles: 360°/3, 360°/4, and so on, up to 360°/12.

6. How can you get the mirrors to form each of the angles you listed in Problem 5 without using a protractor or pattern blocks? (This may not really be possible for the smallest angles. **Hint:** Look at the reflections of the mirrors themselves.)

Name(s) ______________________

7. Which of the angles from Problem 5 yield only mirror-symmetric designs when a pattern block is placed between the mirrors? Which angles can yield either symmetric or asymmetric designs?

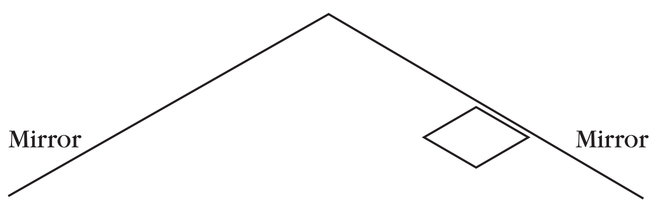
   Symmetric only:

   Symmetric or asymmetric:

8. On a separate sheet of paper, create your own designs using one or more pattern blocks and the hinged mirrors. Record your designs with the help of the template. For each design, make a note about the angle you used and whether your design is symmetric.

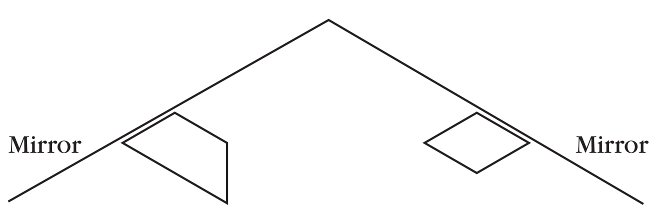
Discussion

A. Set up the mirrors at an angle of 120°, and place a blue pattern block near one mirror and far from the other, as shown on the figure below. Note that, depending on where you are, you will see two, three, or four blocks (including the original and its reflections). Write 2, 3, and 4 on the figure to indicate where you would look from to see that many blocks.



B. Compare the results of using even and odd numbers for *n* when using the angle 360°/*n* between the mirrors. Discuss both the reflections of the mirrors themselves and the reflections of other objects.

C. Note that, when you use an odd value for *n,* the middle reflections of the mirrors create the illusion of a two-sided mirror, in the sense that if you look at them from the left or the right you will see a mirror. However, if you look from a point that is equidistant from the two real mirrors, you may see different objects on the left and right of that middle virtual mirror! (Try the setup shown on the figure below.)



Name(s) ______________________

D. If you allow yourself to stand anywhere, what range of angles allows you to see three copies of a block (including the original)? Four copies? Five?

E. Look at the reflection of your face in the mirrors when they make a 90° angle. Wink with your right eye. What happens in the reflection? Can you explain how and why this is different from your reflection in one mirror?

F. Are some of the reflections you see in the hinged mirrors reflections of reflections? Reflections of reflections of reflections? If so, which ones?

G. Write the letter F and look at its reflections in the hinged mirrors. Which copies of it are identical to the original and which are backward?

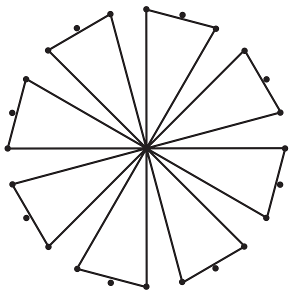
LAB 5.5

Name(s) ________________________

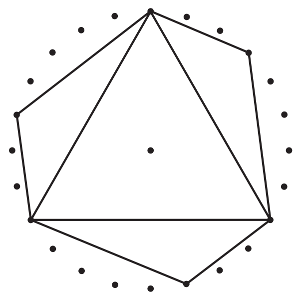
Rotation Symmetry

■ **Equipment:** Circle geoboards, Circle Geoboard Paper

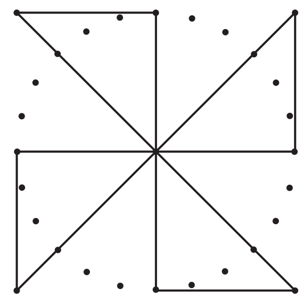
1. Make these designs on your circle geoboards.



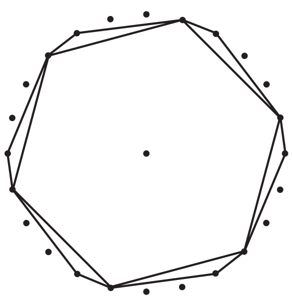
a



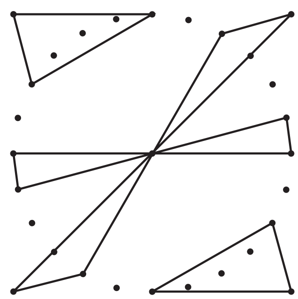
b



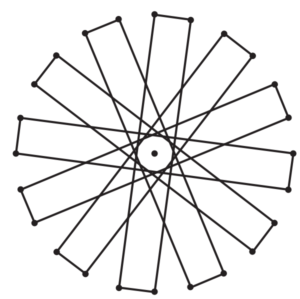
c



d



e



f

Each of the designs has rotation symmetry: If you rotate the geoboard around its center, the design returns to its initial position before you have done a full turn. For example, figure *b* above would look the same after one-third of a full turn. It returns to its initial position three times as you turn it (counting the final time after the full 360° turn). The design is said to have 3-fold rotation symmetry.

2. Each design in Problem 1 has *n*-fold symmetry for some *n*. What is *n* in each case?

a. _______ **b.** _______ **c.** _______ **d.** _______ **e.** _______ **f.** _______

3. Which of the designs in Problem 1 also have line symmetry? How many lines of symmetry does each one have?

4. On a separate sheet of paper, make designs with *n*-fold rotation symmetry for various values of *n*. Some should have line symmetry and some not. Record and label your designs.

Discussion

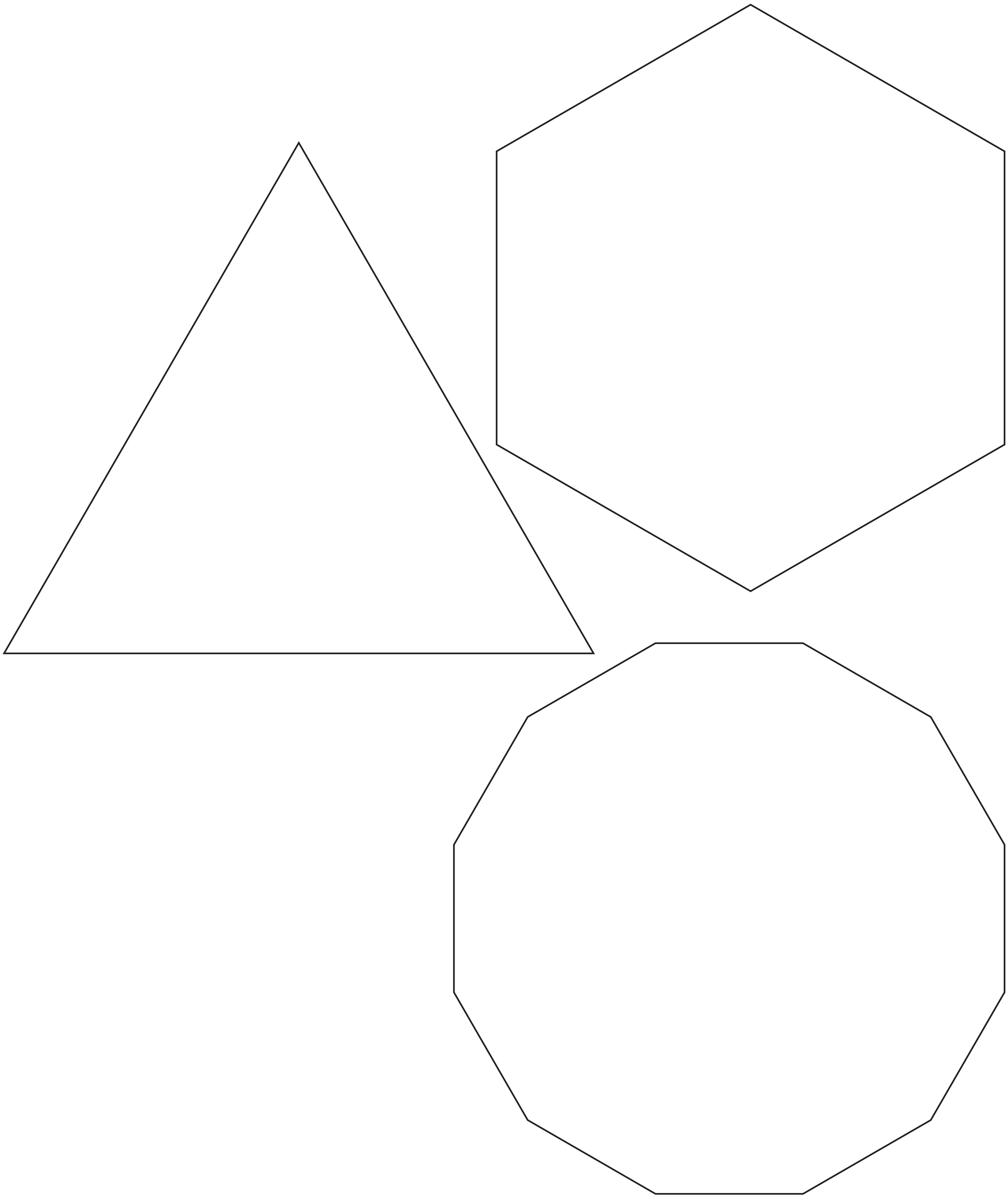
A. It is possible to make the mirror image design for each of the designs in Problem 1. In some cases, the mirror image design will be different from the original one. It is said to have different handedness. Which of the designs have different handedness from their mirror images?

B. Is it possible to have *n*-fold rotation symmetry for any number *n*? What would 1-fold rotation symmetry mean? What about 0-fold? 2.5-fold? –2-fold?

LAB 5.6
Rotation and Line Symmetry

Name(s) ______________________

■ **Equipment:** Pattern blocks, template



Cover the figures on the previous page with pattern blocks. You may use any number and any type of pattern blocks. Each time, draw the resulting figure with the template, and label it with its symmetry properties: If it has rotation symmetry, how many fold? Does it have line symmetry? As you find the various solutions, check them on the table below. If you can make a nonsymmetric design, check the box for 1-fold rotation symmetry. If there is no solution, put an X in the table.

	Triangle		Hexagon		Dodecagon	
	Rot. sym. only	Line sym.	Rot. sym. only	Line sym.	Rot. sym. only	Line sym.
1-fold						
2-fold						
3-fold						
4-fold						
5-fold						
6-fold						
7-fold						
8-fold						
9-fold						
10-fold						
11-fold						
12-fold						

Discussion

A. Which symmetry types are impossible on each of the polygons? Why?

B. What is the relationship between the number of sides of a regular polygon and the possible *n*-fold rotation symmetries once it is covered with pattern blocks? Explain.

C. Describe some strategies for finding new designs from old ones.

D. Given a pattern block figure with a certain symmetry, experiment with making it more symmetric or less symmetric by switching blocks in and out. Keep track of your strategies on a separate sheet of paper.

E. What symmetries do you think would be possible if you tried to cover a figure in the shape of the blue pattern block, with each side equal to 3 inches? Explain your prediction.

LAB 5.7

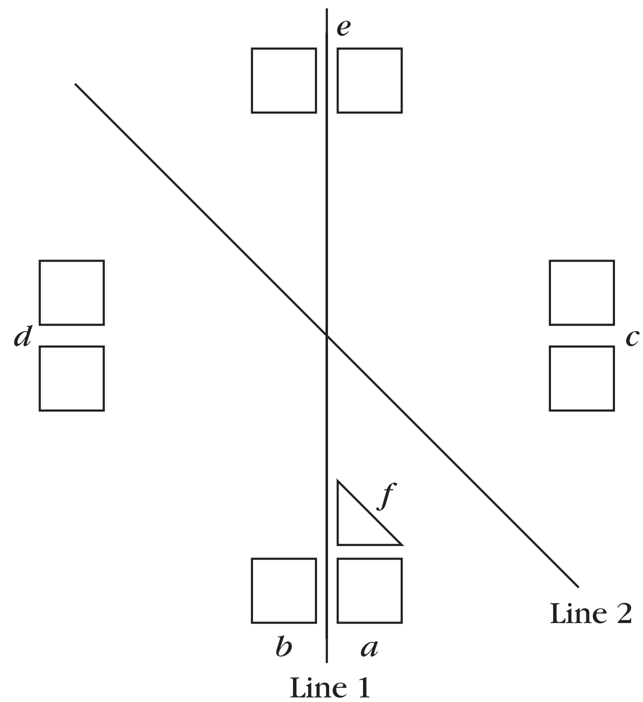
Name(s) ______________________

Two Intersecting Lines of Symmetry

- **Equipment:** Unlined, graph, or dot paper, template

In this activity, you will create designs based on two intersecting lines of symmetry.

Example: Line 1 and line 2 below form a 45° angle. I started my design by tracing a square in position *a*. In order for the lines to be lines of symmetry of the final figure, I had to add several more squares: *a,* reflected in line 1, gave me *b;* *a* and *b,* reflected in line 2, gave me *c*; *c,* reflected in line 1, gave me *d*; *d,* reflected in line 2, gave me *e*.



I ended up with eight squares. Any further reflections get me back to previously drawn squares. At that point, I added a triangle (*f*).

1. **a.** Add triangle reflections to the figure, using the lines of symmetry to guide you.
   b. Add anything else you want to the figure, taking care to preserve the symmetry.
2. Create your own designs, using the following angles between the lines of symmetry. (Use the back of this sheet and additional sheets of paper as necessary.)
   a. 90°
   b. 60°
   c. 45°
   d. 30°

3. Create your own designs, using the following angles between the lines of symmetry.

a. 72°

b. 40°

c. 36°

4. Which angles from Problem 3 yield the same designs as each other? Why?

Discussion

A. For each angle, how many copies of any item in the figure are there in the whole figure? (Include the original in the count.)

B. How is this lab similar to the one with the hinged mirror? How is it different?

C. Is it possible to place a physical object (such as a plastic tangram) right on one of the lines of symmetry and have it be part of the design without ruining the symmetry? If not, why not? If yes, how should it be placed? What else would have to be done?

D. Would you answer Question C differently if you were *drawing* an initial shape in such a way that it overlapped a line of symmetry (as opposed to placing a physical object there)?

E. How are a given shape and the reflection of its reflection related? (Reflect first in one, then in the other line of symmetry.) Do they have the same handedness? How could you obtain one shape from the other?

F. While creating these designs, does it make sense to reflect the lines of symmetry themselves? Would it change the final result? If so, how? If not, how does it change the process of getting to the final result?

LAB 5.8

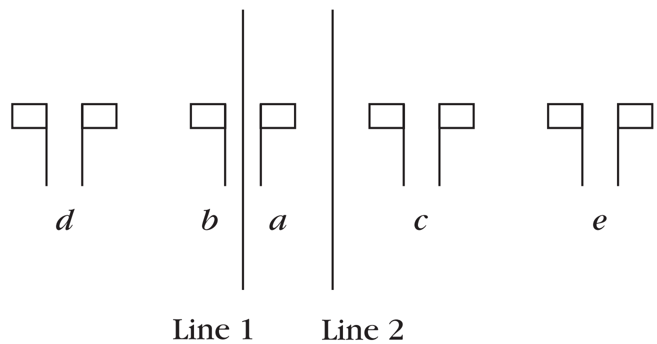
Name(s) ______________________

Parallel Lines of Symmetry

- **Equipment:** Unlined, graph, or dot paper, templates

In this activity, you will create designs based on two parallel lines of symmetry.

Example: In the figure below, I started with one flag in position *a*. In order for bothe lines to be lines of symmetry of the design, I had to draw several more flags. By reflecting *a* in line 1, I got *b*. By reflecting both of them in line 2, I got *c*. By reflecting *c* in line 1, I got *d*. By reflecting *d* in line 2, I got *e*.



1. Add something to the figure above, taking care to preserve the symmetry.
2. Does this process of adding to the figure ever end? If so, when? If not, why not?
3. On a separate sheet of paper, create your own design based on parallel lines of symmetry.
4. Create your own design based on two parallel lines of symmetry and a third line of symmetry perpendicular to the first two.

The remaining problems are very challenging. Work them out on a separate sheet of paper.

5. Create your own design, based on two parallel lines of symmetry and a third line of symmetry at the following angles.
   - **a.** 60°
   - **b.** 45°
   - **c.** 30°
6. Start with a right isosceles triangle. Extend its sides so they are infinite lines of symmetry, and build a design around them.
7. Repeat Problem 6, starting with the following triangles.
   - **a.** Equilateral triangle
   - **b.** Half-equilateral triangle

Discussion

A. How could you simulate the pattern of parallel lines of symmetry in the real world?

B. How are a given shape and the reflection of its reflection related? (Reflect first in one, then in the other line of symmetry.) Do they have the same handedness? How could you obtain one shape from the other?

C. While creating these designs, does it make sense to reflect the mirrors themselves? Would it change the final result? If yes, how? If not, how does it change the process of getting to the final result?

D. Describe the fundamental difference between adding a third mirror at 90° and adding it at another angle.

E. Why are the "mirror triangles" in Problems 6 and 7 included in a lesson on parallel lines of symmetry?