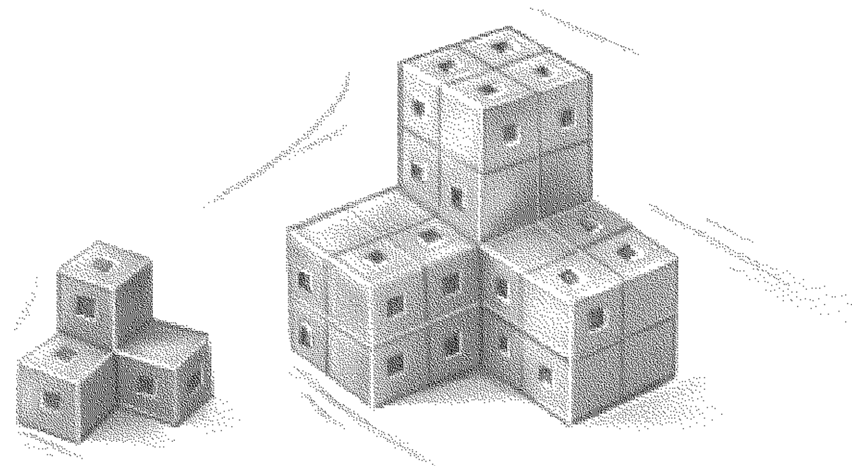
10 Similarity and Scaling



Similarity and scaling are intimately related: By scaling a figure, we obtain another one similar to the original. I have found that it is much easier for students to understand similarity when it is connected with scaling, perhaps because they are more comfortable with multiplication than with ratios.

Two other features of the approach in this section help students get a handle on similarity.

- The work on a Cartesian grid helps students visualize what happens when a figure is scaled.
- The use of the geoboard lattice or the whole numbers of squares in polyominoes helps by providing simpler numbers to work with initially.

In Lab 10.2 (Similar Rectangles), I make a connection between similarity and slope. It is an unfortunate consequence of the artificial boundaries between our algebra and geometry courses that such connections are not made more often.

The other central idea of this section is the relationship between the ratio of similarity and the ratio of areas. This idea is very difficult, as it stands at the intersection of two profound concepts: proportionality and dimension. It certainly cannot be grasped from a brief explanation. In my experience, the most effective way to get it across to students is to explicitly separate out the two dimensions, which we do in Lab 10.1 (Scaling on the Geoboard) and Lab 10.3 (Polyomino Blowups).

See page 222 for teacher notes to this section.

LAB 10.1

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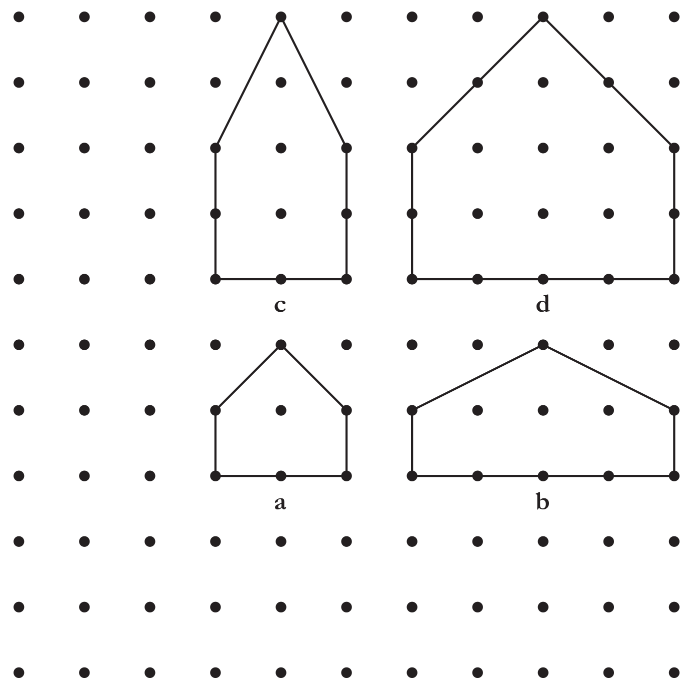
Scaling on the Geoboard

■ **Equipment:** 11 × 11 geoboard, dot paper



The figure shows four houses. House **a** is the original, and the others have been copied from it following very exact rules.

1. What is the rule that was used for each copy?

2. Among the three copies, two are distorted, and one is a scaled image of the original.

a. Which one is scaled?

b. How would you describe the distortions?

When two figures are scaled images of each other, they are said to be *similar*. Similar figures have *equal angles* and *proportional sides*. The sides of one figure can be obtained by multiplying the sides of the other by one number called the *scaling factor*.

3. What is the scaling factor that relates the two similar houses in the figure above?

4. Make a geoboard triangle such that the midpoint of each side is on a peg. Make your triangle as different as possible from those of your neighbors. Join the midpoints with another rubber band, making a smaller triangle.
   - **a.** Describe the resulting figure, paying attention to equal segments and angles, congruent triangles, parallel lines, and similar figures.
   - **b.** Find the slopes of the lines you believe to be parallel, the length of the segments you believe to be equal, and the scaling factor for the figures you believe to be similar.
5. Make a geoboard quadrilateral such that the midpoint of each side is on a peg. Make your quadrilateral as different as possible from those of your neighbors. Stretch another rubber band around consecutive midpoints, making a smaller quadrilateral.
   - **a.** Describe the resulting figure, paying attention to equal segments and angles and parallel lines.
   - **b.** Find the slopes of the lines you believe to be parallel and the length of the segments you believe to be equal.

Discussion

A. The rules discussed in Problem 1 can be expressed in terms of coordinates (assuming the origin is at the bottom left peg) or in terms of measurements. Explain.

B. For any pair of similar figures, there are two scaling factors relating them. How are the factors related to each other?

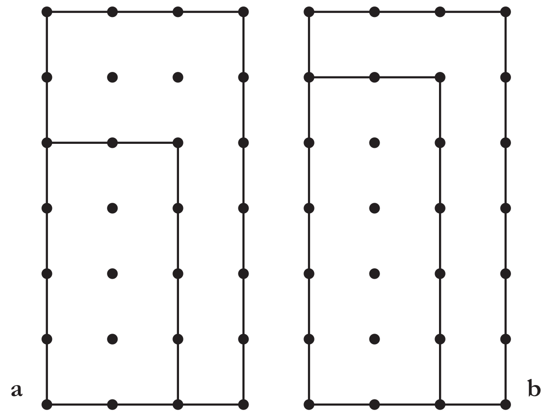
C. Try to make a triangle so that two of its sides have midpoints on pegs, but the third does not. What happens? Explain.

D. Problems 4 and 5 are special cases of general theorems that apply to the midpoints of *any* triangle or quadrilateral. State the theorems. Explain how the result in Problem 5 is a consequence of the result in Problem 4.

LAB 10.2
Similar Rectangles

Name(s) ____________________

■ **Equipment:** 11 × 11 geoboard



The figure above shows geoboard rectangles nested inside each other.

1. Explain why the two rectangles on the left are similar but the two rectangles on the right are not.

2. Draw the diagonal from the bottom left to the top right of the larger rectangle in **a**. Note that it goes through the top right vertex of the smaller rectangle. Repeat on **b**. How is it different?

3. The rectangles in **a** are part of a family of similar geoboard rectangles. If we include only rectangles with a vertex at the origin (the bottom left peg of the geoboard), the family includes ten rectangles (five horizontal ones and five vertical ones—two of the vertical ones are shown in the figure).

   a. List the rectangles in that family by listing the coordinates of their top right vertex.

   b. What is the slope of the diagonal through the origin for the vertical rectangles?

   c. What is the slope of the diagonal for the horizontal rectangles?

4. Including the one in Problem 3, there are ten families of similar rectangles (including squares) on the geoboard. Working with your neighbors, find them all, and list all the rectangles in each family. (Again, assume a vertex at the origin, and use the coordinates of the top right vertex for the list. Only consider families with more than one rectangle.)

5. Working with your neighbors, find every geoboard slope between 1 and 2. (Express the slopes both as fractions and as decimals.) Counting 1 and 2, there are seventeen different slopes.

Discussion

A. Problem 2 is an example of using the *diagonal test for similar rectangles*. Explain.

B. In Problems 3 and 4, how does symmetry facilitate the job of listing the rectangles?

C. What are the advantages of fractional versus decimal notation in Problem 5?

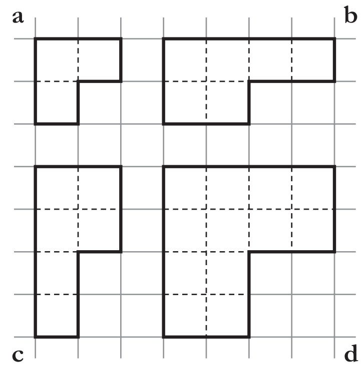
D. Explain how to use the answers to Problem 5 to create lists of geoboard slopes in the following ranges.

- **a.** Between 0.5 and 1
- **b.** Between –1 and –2
- **c.** Between –0.5 and –1

LAB 10.3
Polyomino Blowups

Name(s) ______________________

- **Equipment:** Interlocking cubes, 1-Centimeter Grid Paper, Polyomino Names Reference Sheet



The figure above shows a bent tromino (**a**) and three copies of it. The **b** copy has been doubled *horizontally,* the **c** copy has been doubled *vertically,* and the **d** copy has been doubled in *both* dimensions.

1. Which of the doubled figures is similar to the original? Explain.

2. For each polyomino with area less than 5, draw the polyomino and three copies on grid paper following the pattern in the figure above. As you work, record the perimeter and area of each stretched polyomino in the table.

	Perimeter				**Area**			
		Doubled				**Doubled**		
	Original	**Horiz.**	**Vertic.**	**Both**	**Original**	**Horiz.**	**Vertic.**	**Both**
Monomino								
Domino								
Bent								
Straight								
Square								
l								
i								
n								
t								

3. Study the table for patterns relating the numbers in the different columns. Write down what you observe.

4. The most important patterns are the ones relating the original figure with the similar figure. State the patterns for perimeter and area.

5. Write down a prediction about what may happen to perimeter and area if you triple, and quadruple, a polyomino in both dimensions.

6. Test your prediction by drawing a few polyominoes and their tripled and quadrupled copies. Write down your conclusions.

The following problems are puzzles involving similar polyominoes.

7. **a.** With your interlocking cubes, make tiles in the shape of the **l** and **t** tetrominoes.
   b. On grid paper, draw the five doubled tetrominoes (doubled in both dimensions!).
   c. Use your tiles to cover each doubled figure. Record your solutions.
8. Repeat Problem 7 with the following shapes, being sure to multiply the dimensions both horizontally and vertically.
   a. Tripled tetrominoes, using **l** and **t** tiles
   b. Doubled pentominoes, using P and N tiles
   c. Tripled pentominoes, using P and L tiles

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Discussion

A. What is the relationship between the scaling factor and the ratio of perimeters?

B. What is the relationship between the scaling factor and the ratio of areas? Why is this answer different from the answer to Question A?

C. How many tetrominoes does it take to tile a tripled tetromino? How many pentominoes does it take to tile a tripled pentomino? Explain.

D. How many polyominoes does it take to tile a polyomino whose area has been multiplied by k? Explain.

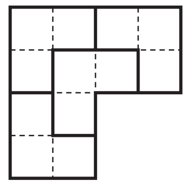
LAB 10.4

Name(s) ____________________

Rep-Tiles

Equipment: Interlocking cubes, 1-Centimeter Grid Paper, Polyomino Names Reference Sheet, template

A shape is a rep-tile if it can be used to tile a scaled copy of itself. For example, the bent tromino is a rep-tile, as you can see in the figure below.



1. The **l** tetromino and the P pentomino are rep-tiles. Use cubes and grid paper to tile the doubled and tripled versions of the shapes with the original. Keep a record of your solutions.
2. How many original shapes did you need to tile the following?
   a. The doubled figures
   b. The tripled figures
3. Which other polyominoes of area less than or equal to 5 are rep-tiles? Working with your neighbors, find the tilings to support your answer.
4. Classify the pattern blocks into two groups: those that are rep-tiles and those that are not. (Use the template to make your drawings.)
5. Using each triangle on the template, show that it is a rep-tile.
6. In most cases, it takes four copies of a triangle to show it is a rep-tile. Illustrate the following exceptions.
   a. Show that two right isosceles triangles tile a larger right isosceles triangle.
   b. Show that three half-equilateral triangles tile a larger half-equilateral triangle.
7. Classify the quadrilaterals on your template into two groups: those that are rep-tiles and those that are not.

Discussion

A. When a figure is scaled by a factor *k,* its area is multiplied by ____. Explain.

B. Prove that *any* triangle is a rep-tile.

C. What are the scaling factors in Problem 6?

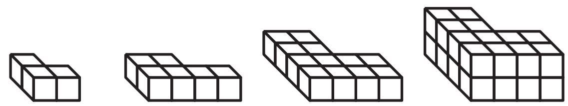
D. Find a triangle that can tile a scaled copy of itself using five tiles.

LAB 10.5
3-D Blowups

Name(s) ____________________

■ **Equipment:** Interlocking cubes

In three dimensions, similar solids are obtained by scaling in all three dimensions. The figure below shows an example, with the scaling happening in three steps: width, depth, and height.



1. The following questions refer to the figure above.

a. Which two solids are similar?

b. What is the scaling factor?

c. What are their surface areas?

d. What is the ratio of the surface areas?

e. What are their volumes?

f. What is the ratio of the volumes?

2. Working with your neighbors, make other pairs of similar solids with your interlocking cubes using a scaling factor of 2, and fill out the table below. Use only a few cubes in the smaller solid!

Surface area			Volume		
Original	**Blowup**	**Ratio**	**Original**	**Blowup**	**Ratio**

Name(s) ____________________

3. Repeat Problem 2 with a scaling factor of 3. Use even fewer cubes in the smaller solid!

Surface area			**Volume**		
Original	**Blowup**	**Ratio**	**Original**	**Blowup**	**Ratio**

4. Choose one of the solids in the figure below, different from your neighbors' choices. Without using physical cubes, find the surface area and volume for blowups of your solid, and fill out the table. You may use sketches.

Scaling factor	**Surface area**	**Area ratio**	**Volume**	**Volume ratio**
1				
2				
3				
4				
5				
6				

Discussion

A. When you use a scaling factor *k*, what happens to each individual unit cube? How does this determine the ratio of surface areas? The ratio of volumes? Explain.

B. Write a formula for the surface area A of a scaled solid as a function of the original surface area A_0 and the scaling factor k. Repeat for the volume.

C. Write a formula for the volume V of the blown-up versions of the solid you studied in Problem 4 as a function of the surface area A and the scaling factor k. Compare your formula with those of your neighbors.

LAB 10.6
Tangram Similarity

Name(s) ______________________

■ **Equipment:** Tangrams

In this lab, write your answers in simple radical form.

1. Fill out this table of tangram perimeters. Part of the first row, where the leg of the small triangle has been taken for the unit of length, has been filled out for you. In the next row, the leg of the medium triangle is the unit, and so on.

Legs			Perimeters				
Sm Δ	**Md Δ**	**Lg Δ**	**Sm Δ**	**Md Δ**	**Lg Δ**	**Square**	**Parallelogram**
1	$\sqrt{2}$		$2+\sqrt{2}$	$2+2\sqrt{2}$		4	
	1						
		1					

2. Fill out this table of tangram areas.

Legs			Areas				
Sm Δ	**Md Δ**	**Lg Δ**	**Sm Δ**	**Md Δ**	**Lg Δ**	**Square**	**Parallelogram**
1	$\sqrt{2}$		$\frac{1}{2}$	1		1	
	1						
		1					

3. The tangram triangles in the tables above are similar. For each pair of triangles, there are two scaling factors (smaller to larger and larger to smaller). Find all such scaling factors.

4. For each answer in Problem 3, there is a corresponding ratio of areas. What is it?

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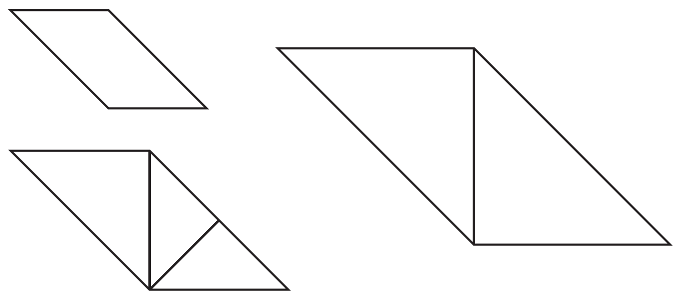
5. The figure below shows three similar tangram figures. Check that the scaling factor from smallest to largest is 2. What are the other scaling factors?



6. **Puzzle:** Find pairs of similar tangram figures using the tangrams from one set only. For each pair, sketch your solution on a separate sheet and record the corresponding scaling factor. Note: Congruent pairs count as similar.

Discussion

A. In the case of tangrams, is it easier to find the ratio of areas or the ratio of similarity of two similar figures? Why?

B. What are some shortcuts for the process of filling out the tables in Problems 1 and 2?

C. Congruent figures have equal angles and proportional sides, so they're also similar. What's the ratio of similarity in congruent figures?

D. In Problem 6, is it possible to find a pair of similar tangram figures that uses all seven pieces?

LAB 10.7

Name(s) ____________________

Famous Right Triangles

	Leg_1	Leg_2	Hypotenuse
a.	1	1	
b.	1	2	
c.	1		2
d.	3	4	
e.		12	13

1. Fill out the table above. If there are square roots, use simple radical form.

Refer to the completed table to solve Problems 2–6.

2. Which triangle is a half-square? Sketch it, indicating the sides and angles.

3. Which triangle is a half-equilateral? Sketch it, indicating the sides and angles.

4. Which triangles have whole-number sides? In such cases, the three numbers are called a *Pythagorean triple*.

5. Find the sides of triangles similar to these five with a scaling factor of 2.

6. Find the sides of triangles similar to these five with a scaling factor of x.

Name(s) ______________________

Solve Problems 7–12 two ways.

a. By the Pythagorean theorem

b. By similar triangles

7. A right triangle with sides of length 10 and 20 is similar to one of the five triangles. What are the possibilities for the third side?

8. A square has side 60. How long is the diagonal?

9. A square has diagonal 30. How long is the side?

10. An equilateral triangle has side 70. What is the height?

11. An equilateral triangle has height 40. How long is the side?

12. A right triangle is also isosceles. One of its sides is 50. What are the possibilities for the other sides?

Discussion

A. Consider pattern blocks, tangrams, and the 11×11 geoboard. Which famous right triangle is most relevant to each?