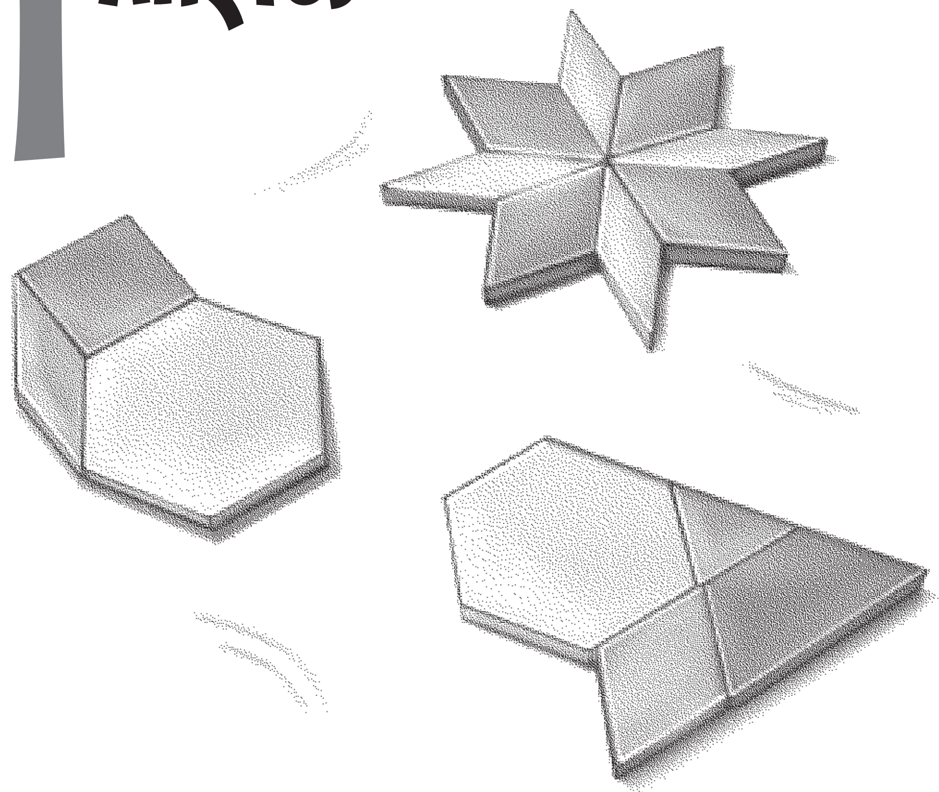
1 Angles



An essential foundation of secondary school geometry is the concept of angle. Many students do not have enough experience with angles when, at the beginning of a geometry class, we tell them how to use a protractor. Often, we are surprised at how difficult this idea is for many students. I have banged my head against that wall, repeating over and over to some students how a protractor is used, and eventually made some headway. But that was a frustrating approach. What I was missing was that students have trouble because it is difficult to measure something when you are not sure what it is. I developed the first lessons in this section in order to build the intuitive foundation necessary for much of the work in a traditional geometry course. Of course, understanding angles is also necessary in reform-minded courses (such as the one I teach at the Urban School of San Francisco) and in later work in trigonometry.

Many geometry textbook authors are concerned about distinguishing angles from their measures, and use a different notation for each. This distinction is rather subtle and, frankly, I do not burden my students with it. There is not much to say about angles unless we are talking about their measures. If you are concerned with inconsistencies between sheets copied from this book and the textbook you use, tell your students that notation and language vary from book to book. It is unlikely to upset them, as long as you make clear what notation you want them to use in tests and other assessments.

I do not believe in the widespread but absurd restriction that angle measurements must be less than 180°. We are talking to some students who can turn 360° on their skateboards, and to some who will soon be studying trigonometric functions. All of them live in a world where polygons can be nonconvex (consider the Star Trek® insignia), and therefore include angles greater than 180°. Angles whose measures are between 180° and 360° are called *reflex* angles. You may add this word to your students' vocabulary, along with the better-known *acute, right, obtuse,* and *straight,* even if the word is not in the textbook you use. One of the theorems we commonly teach in geometry (and in this book) is about the sum of the angles in a polygon. The theorem still holds in the case of nonconvex polygons. Why limit this result with the artificial constraint that angles cannot measure more than 180°?

One particular aspect of the study of angles is the study of central and inscribed angles in a circle. This is an interesting topic, but difficult, and is usually relegated to the very end of textbooks. Consequently, some of us never get to it, and the designers of standardized tests use this fact to separate students in honors classes from the rest of the population. The activities in the second part of this section are intended to give students in all sorts of geometry classes a jump-start on the ideas involved. You may teach these early in the year and review them later when you reach that point in your textbook; alternatively, you may save them for later and teach them right before they arise in the textbook. However, if there is a risk that you may not get to that point, you may consider moving that whole topic to an earlier time in your course plan.

I ended this section with a related topic: "soccer angles." These lessons are introduced with a motivating real-world problem and lead to a tough optimization challenge: Where is the best place on the field to shoot at the goal from?

See page 167 for teacher notes to this section.

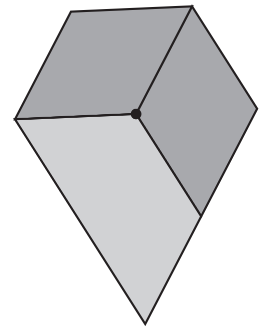
LAB 1.1

Name(s) ______________________________

Angles Around a Point

Equipment: Pattern blocks

Place pattern blocks around a point so that a vertex (corner) of each block touches the point and no space is left between the blocks. The angles around the point should add up to exactly 360°.



For example, with two colors and three blocks you can make the figure at right.

Use the chart below to keep track of your findings.

- Every time you find a new combination, circle the appropriate number on the list below.
- Cross out any number you know is impossible.
- If you find a possible number that is not on the list, add it.

Since the two-colors, three-blocks solution is shown above, it is circled for you.

Colors:	How many blocks you used:								
all blue	3	4	5	6					
all green	3	4	5	6					
all orange	3	4	5	6					
all red	3	4	5	6					
all tan	3	4	5	6					
all yellow	3	4	5	6					
two colors	3	4	5	6	7	8	9	10	11	12
three colors	3	4	5	6	7	8	9	10	11	12
four colors	3	4	5	6	7	8	9	10	11	12
five colors	3	4	5	6	7	8	9	10	11	12
six colors	3	4	5	6	7	8	9	10	11	12

How many solutions are there altogether? __________

Discussion

A. Which blocks offer only a unique solution? Why?

B. Why are the tan block solutions only multiples of 4?

C. Explain why the blue and red blocks are interchangeable for the purposes of this activity.

D. Describe any systematic ways you came up with to fill in the bottom half of the chart.

E. How do you know that you have found every possible solution?

F. Which two- and three-color puzzles are impossible, and why?

G. Which four-color puzzles are impossible, and why?

H. Why is the five-color, eight-block puzzle impossible?

I. Which six-color puzzles are impossible, and why?

LAB 1.2
Angle Measurement

Name(s) ______________________________

■ **Equipment:** Pattern blocks, template

1. What are the measures of the angles that share a vertex at the center of

a. A Chrysler symbol? ____________

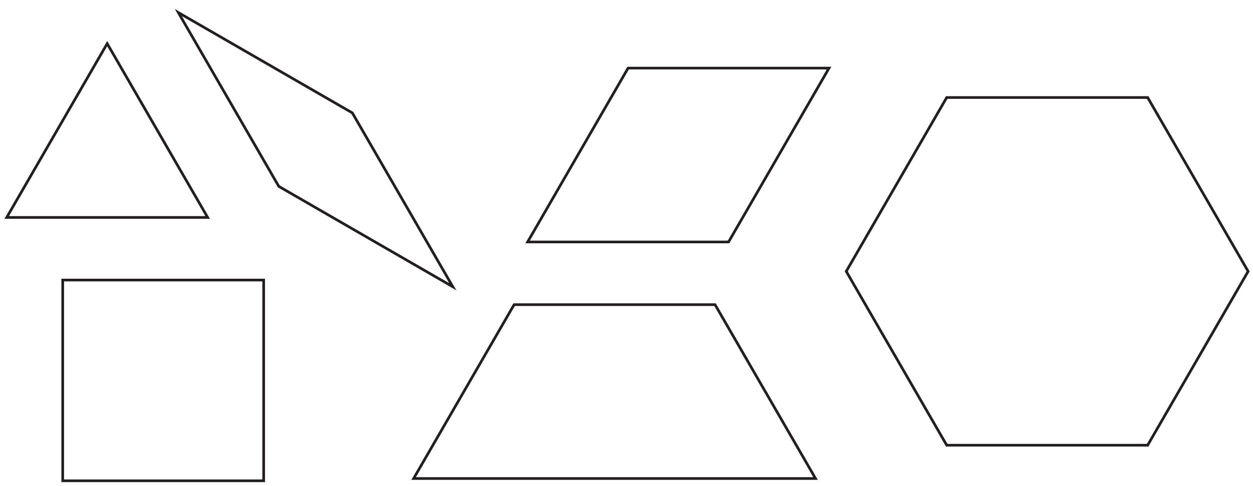
b. A Mercedes symbol? ____________

c. A peace sign? ____________

d. A clock, between consecutive hours? ____________

e. A cross? ____________

2. Find the measures of all the angles for each of the pattern blocks shown below. Write the angle measures in these shapes.

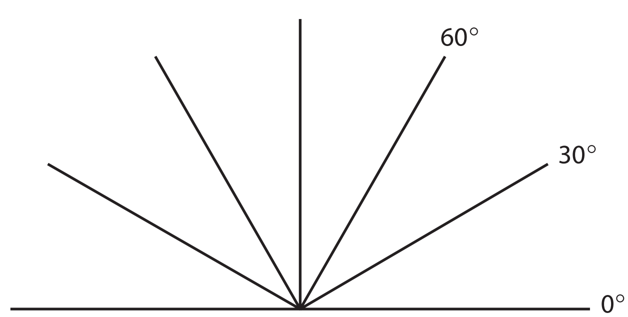


3. One way to measure angles is to place smaller angles inside larger ones. For example, six copies of the small angle on the tan pattern block fit inside the figure below. This figure, called a **protractor**, can be used to measure all pattern block angles.

a. Mark the rest of the lines in the figure with numbers.

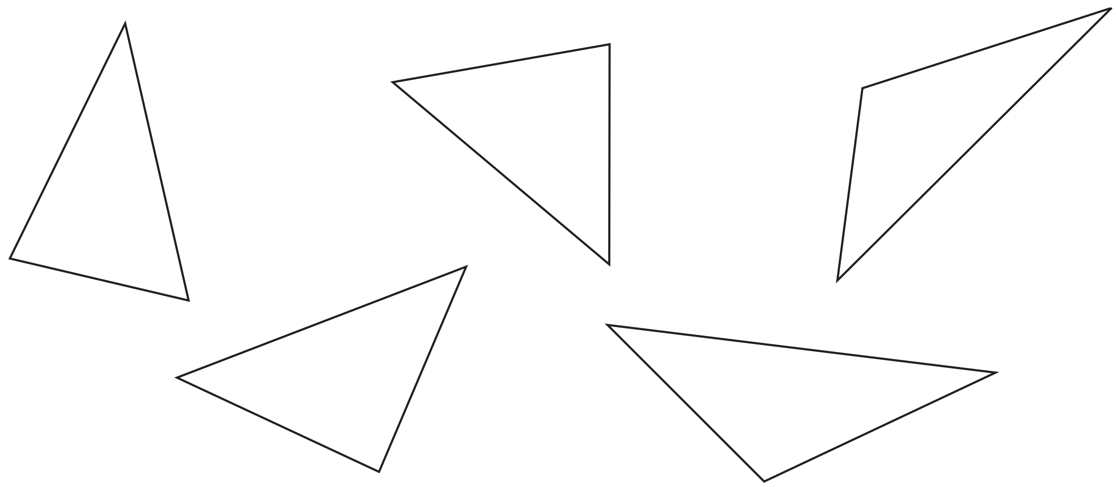
b. Use it to check the measurements of the pattern block angles.

c. Using the tan pattern block, add the 15° lines between the 30° lines shown on the protractor.



Name(s) ______________________

4. Use the protractor on your template to measure the angles in the triangles below. For each one, add up the angles. Write the angle sum inside each triangle.



5. Use the protractor on your template to draw the following angles in the space below.

a. 20°

b. 50°

c. 100°

Discussion

Which of the questions in Problem 1 did you find most difficult to answer? What made the others easier?

LAB 1.3
Clock Angles

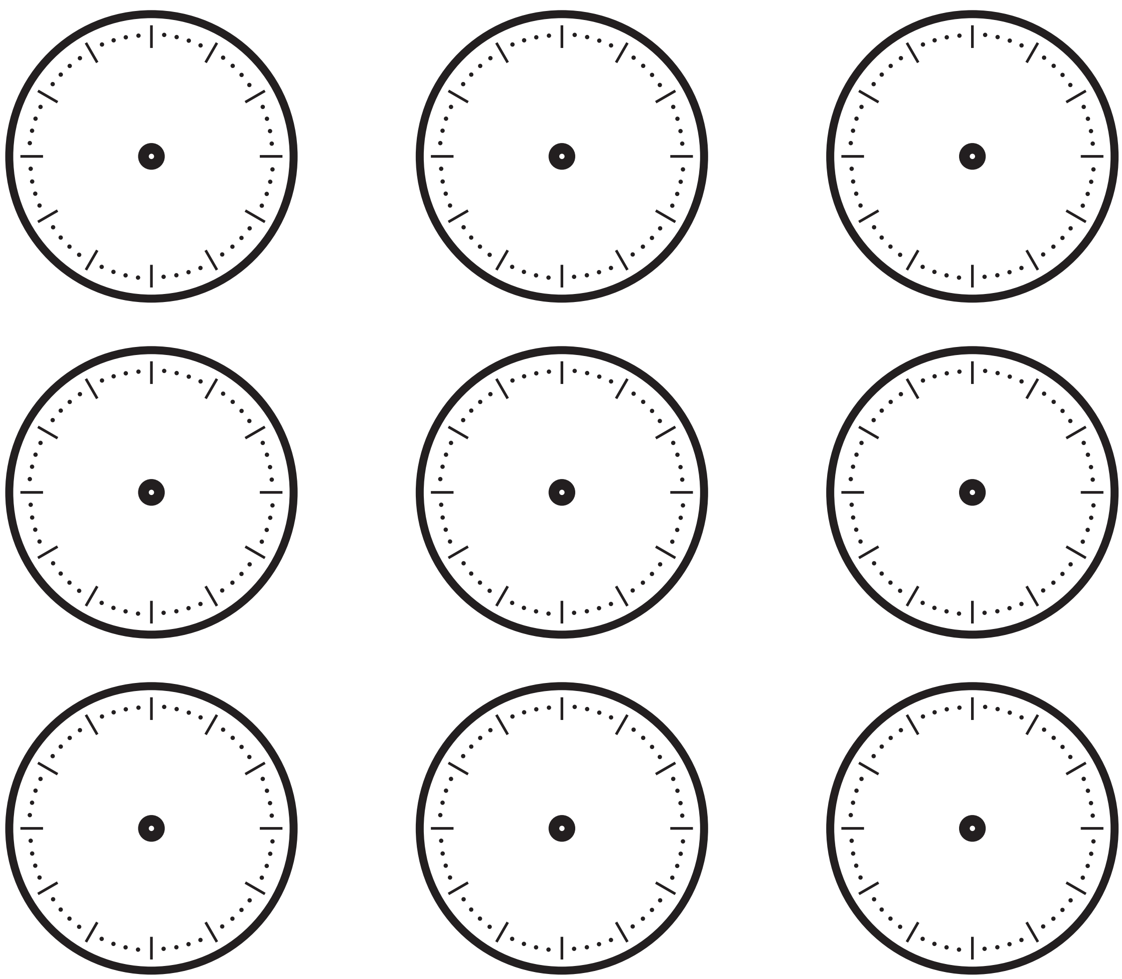
Name(s) ______________________

What angles do the hour and minute hands of a clock make with each other at different times? Write an illustrated report.

- Start by figuring it out on the hour (for example, at 5:00).
- Then see if you can answer the question for the half-hour (for example, at 5:30).
- Continue exploring increasingly difficult cases.

Remember that the hour hand moves, so, for example, the angle at 3:30 is *not* 90° but a little less, since the hour hand has moved halfway toward the 4.

Use the clocks below for practice. Cut and paste clocks into your report.



Discussion

A. How many degrees does the hour hand travel in an hour? In a minute?

B. How many degrees does the minute hand travel in an hour? In a minute?

LAB 1.4
Angles of Pattern Block Polygons

Name(s) ____________________________

■ **Equipment:** Pattern blocks

1. Find the *sum* of the angles for each pattern block, and record them below.
   - **a.** Green __________
   - **b.** Orange __________
   - **c.** Blue __________
   - **d.** Tan __________
   - **e.** Red __________
   - **f.** Yellow __________
2. Which blocks have the same sum?
3. Use pattern blocks to make a polygon such that the sum of its angles is the same as the sum for the hexagon. Sketch your solution in the space at right. How many sides does it have?
4. Use pattern blocks to make a polygon such that the sum of its angles is less than the sum for the hexagon but more than the sum for the square. Sketch your solution in the space at right. How many sides does it have?
5. Use pattern blocks to make a polygon such that the sum of its angles is greater than the sum for the hexagon. Sketch your solution in the space at right. How many sides does it have?
6. For each number of sides from 3 to 12, make a pattern block polygon and find the sum of its angles. Sketch your solutions on a separate sheet of paper and fill out the table below.

Sides	Sum of the angles
3	
4	
5	
6	
7	

Sides	Sum of the angles
8	
9	
10	
11	
12	

7. If a pattern block polygon had 20 sides, what would be the sum of its angles? Explain.

8. If the sum of the angles of a pattern block polygon were 4140°, how many sides would it have? Explain.

9. If the sum of the angles of a pattern block polygon were 450°, how many sides would it have? Explain.

10. What is the relationship between the number of sides of a pattern block polygon and the sum of its angles? Write a sentence or two to describe the relationship, or give a formula.

LAB 1.5

Name(s) ____________________

Angles in a Triangle

- **Equipment:** Template

Types of triangles

Obtuse: contains one obtuse angle
Right: contains one right angle
Acute: all angles are acute

1. Could you have a triangle with two right angles? With two obtuse angles? Explain.

2. For each type of triangle listed below, give two possible sets of three angles. (In some cases, there is only one possibility.)
   - **a.** Equilateral: ____________, ____________
   - **b.** Acute isosceles: ____________, ____________
   - **c.** Right isosceles: ____________, ____________
   - **d.** Obtuse isosceles: ____________, ____________
   - **e.** Acute scalene: ____________, ____________
   - **f.** Right scalene: ____________, ____________
   - **g.** Obtuse scalene: ____________, ____________

3. If you cut an equilateral triangle exactly in half, into two triangles, what are the angles of the "half-equilateral" triangles? ____________

4. Which triangle could be called "half-square"? ____________

5. Explain why the following triangles are impossible.
   - **a.** Right equilateral

   - **b.** Obtuse equilateral

Name(s) ____________________

6. Among the triangles listed in Problem 2, which have a pair of angles that add up to 90°?

7. Make up two more examples of triangles in which two of the angles add up to 90°. For each example, give the measures of all three angles.

____________, ____________

8. Complete the sentence:
"In a right triangle, the two acute angles . . . "

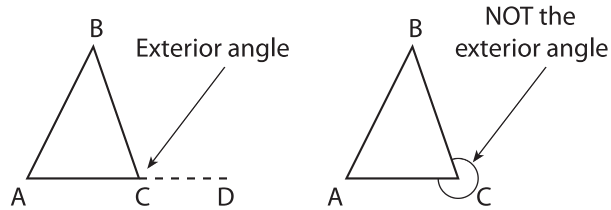
9. Trace all the triangles on the template in the space below and label them by type: equilateral (EQ), acute isosceles (AI), right isosceles (RI), obtuse isosceles (OI), acute scalene (AS), right scalene (RS), half-equilateral (HE), obtuse scalene (OS).

LAB 1.6

Name(s) ____________________

The Exterior Angle Theorem

In this figure, $\angle A = 60°$ and $\angle B = 40°$. One side of $\angle C$ has been extended, creating an *exterior angle:*



1. Without measuring, since this figure is not to scale, find the interior angle at C ($\angle ACB$) and the exterior angle at C ($\angle BCD$).

Interior angle at C __________

Exterior angle at C __________

2. Make up two other examples of triangles where two of the angles add up to 100°. Write all three angles for each triangle in the spaces below. Sketch the triangles in the space at right.

______________, ______________

3. For each of your triangles, find the measures of all three exterior angles.

Triangle one: ________________

Triangle two: ________________

4. In the space at right, sketch an example of a triangle ABC where the exterior angle at B is 65°.

a. What is the measure of the interior angle at B?

____

b. What is the sum of the other two interior angles ($\angle A$ and $\angle C$)?

____

5. Repeat Problem 4 with another example of a triangle ABC with a 65° exterior angle at B.

a. What is the interior angle at B?

____

b. What is the sum of the other two interior angles ($\angle A$ and $\angle C$)?

____

6. I am thinking of a triangle *ABC*. Exterior angle *A* is 123°. Interior angle *B* is given below. What is the *sum* of interior angles *B* and *C*? (Use a sketch to help you work this out.)

a. If $\angle B = 10°$ __________

b. If $\angle B = 20°$ __________

c. Does it matter what $\angle B$ is? Explain.

7. I am thinking of a triangle *ABC*. Exterior angle *A* is $x°$. What is the sum of interior angles *B* and *C*? Explain how you get your answer.

8. Complete this statement and explain why you think it is correct:

The Exterior Angle Theorem: An exterior angle of a triangle is always equal to . . .

9. What is the sum of the two acute angles in a right triangle? Is this consistent with the exterior angle theorem? Explain. (**Hint:** What is the exterior angle at the right angle?)

10. What is the sum of all three exterior angles of a triangle? Find out in several examples, such as the ones in Problems 1, 2, 4, and 5. Explain why the answer is always the same.

Discussion

A. How many exterior angles does a triangle have?

B. What is the sum of an interior angle and the corresponding exterior angle?

C. Draw a triangle, and extend all the sides in both directions. Mark the interior angles with an *i,* the exterior angles with an *e,* and the angles that are vertical to the interior angles with a *v*.

D. In Problems 4–6, are the triangles discussed acute, right, or obtuse? How do you know?

E. If a triangle *ABC* has an exterior angle *A* of 50° and an interior angle *B* of $x°$, what is the interior angle *C* in terms of x? What is the sum of angles *B* and *C* in terms of x?

LAB 1.7

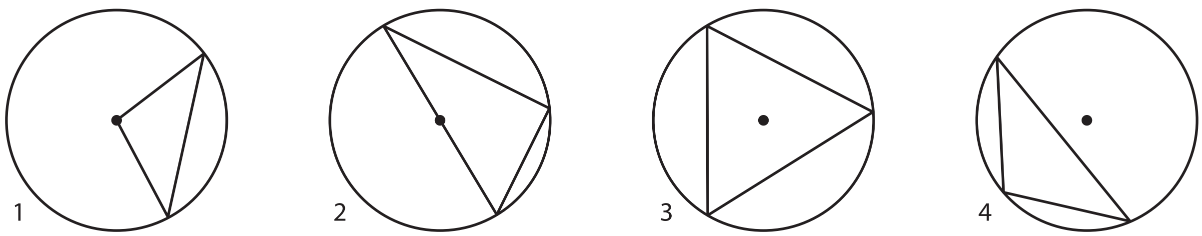
Name(s) ______________________

Angles and Triangles in a Circle

■ **Equipment:** Circle geoboard, Circle Geoboard Paper

Types of triangles

Equilateral (EQ)	Acute isosceles (AI)
Right isosceles (RI)	Obtuse isosceles (OI)
Acute scalene (AS)	Right scalene (RS)
Half-equilateral (HE)	Obtuse scalene (OS)



1. Make triangles on the circle geoboard, *with one vertex at the center* and the other two on the circle. (See circle 1 above.)
   - **a.** Make one of each of the eight types of triangles listed above, if possible. You do not have to do them in order!
   - **b.** Sketch one of each of the types of triangles it was possible to make on circle geoboard paper. Identify which type of triangle it is, and label all three of its angles with their measures in degrees. (Do not use a protractor! Use geometry to figure out the angles.)
2. Thinking back:
   - **a.** What is true of all the possible triangles in Problem 1?
   - **b.** Summarize your strategy for finding the angles. Describe any shortcuts or formulas you used.

Definition: A triangle is *inscribed* in a circle if all three of its vertices are on the circle.

3. Repeat Problem 1 with inscribed triangles such that the circle's center is *on a side* of the triangle. (See circle 2.) **Hint:** Drawing an additional radius should help you find the measures of the angles.

4. Thinking back:

a. Summarize your strategy for finding the angles in Problem 3. Describe any shortcuts or formulas.

b. Write the exterior angle theorem. How can it be used to help find the angles?

5. What is true of all the possible triangles in Problem 3? Prove your answer.

6. Repeat Problem 1 with inscribed triangles such that the circle's center is *inside* the triangle. (See circle 3.)

7. Repeat Problem 1 with inscribed triangles such that the circle's center is *outside* the triangle. (See circle 4.)

Discussion

A. Which triangles are impossible in Problem 1? Why?

B. In Problem 1, what is the smallest possible angle at the vertex that is at the circle's center? Explain.

C. Explain why the angle you found in Question B is the key to Problem 1.

D. How do you use the isosceles triangle theorem to find the other two angles of the triangles in Problem 1?

E. What is true of all the triangles in Problem 6? All the triangles in Problem 7? Explain.

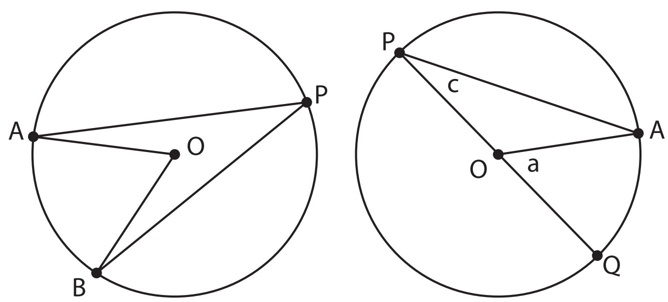
LAB 1.8
The Intercepted Arc

Name(s) ______________________________

Definitions:

A *central angle* is one with its vertex at the center of the circle.
An *inscribed angle* is one with its vertex on the circle.
The arc *intercepted* by an angle is the part of the circle that is inside the angle.
The *measure of an arc* in degrees is the measure of the corresponding central angle.

1. Use the definitions above to fill in the blanks: In the figure below left, $\angle APB$ is ____________, $\angle AOB$ is ____________, and $\widehat{AB}$ is ______________________________.



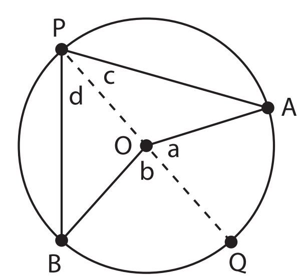
2. In the figure above right, which arc is intercepted by $\angle AOQ$? __________
Which arc is intercepted by $\angle APQ$? __________

3. If $\angle a = 50°$, what is $\angle c$? Explain.

4. In general, what is the relationship between $\angle a$ and $\angle c$? Explain.

5. In the figure at the right, if $\angle AOB = 140°$ and $\angle a = 50°$, what is $\angle b$? What is $\angle c$? What is $\angle d$? What is $\angle APB$?

6. Repeat Problem 5 with $\angle AOB = 140°$ and three different values for $\angle a$. Arrange your results in a table.



7. What is the relationship between $\angle APB$ and $\angle AOB$? Explain.

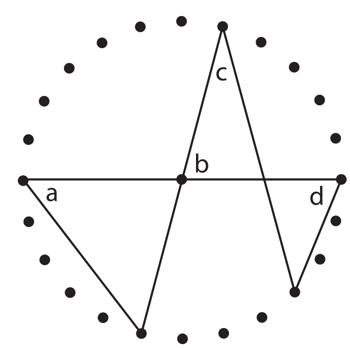
8. Write a sentence about the relationship between an inscribed angle and the corresponding central angle. If you do this correctly, you have stated the inscribed angle theorem.

9. On a separate sheet, use algebra to prove the inscribed angle theorem for the case illustrated in Problem 5.

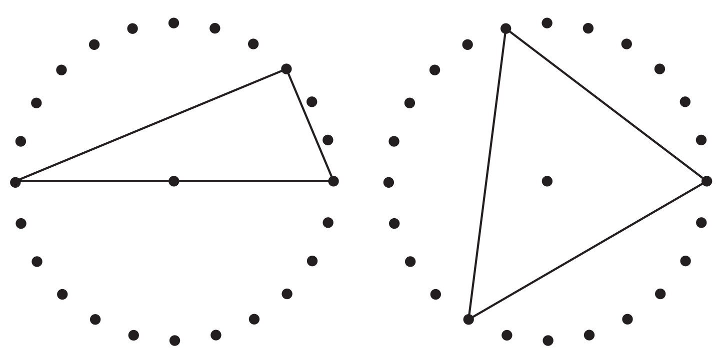
10. There is another case for the figure in Problem 5, where O is outside of $\angle APB$. On a separate sheet, draw a figure and write the proof for that case.

Discussion

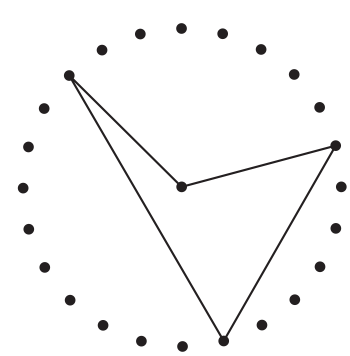
A. Find the measure of angles *a, b, c,* and *d* in the figure at right. Explain.



B. What are the interior angles in the triangles below?



C. What are the measures of the central angle, the corresponding inscribed angle, and the intercepted arc in the figure at right?

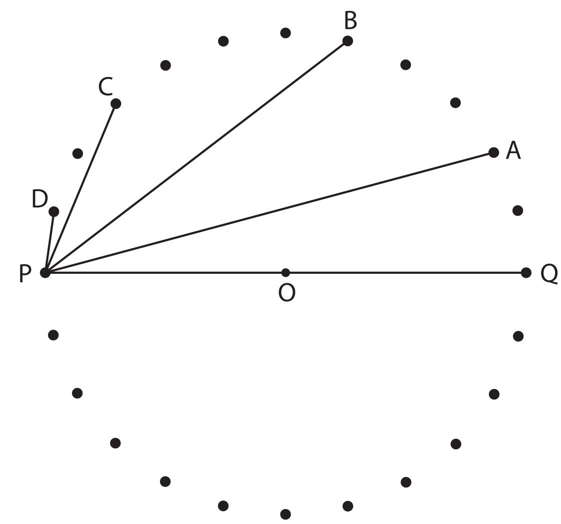


D. Inscribe a triangle in the circle geoboard so that its interior angles are 45°, 60°, and 75°.

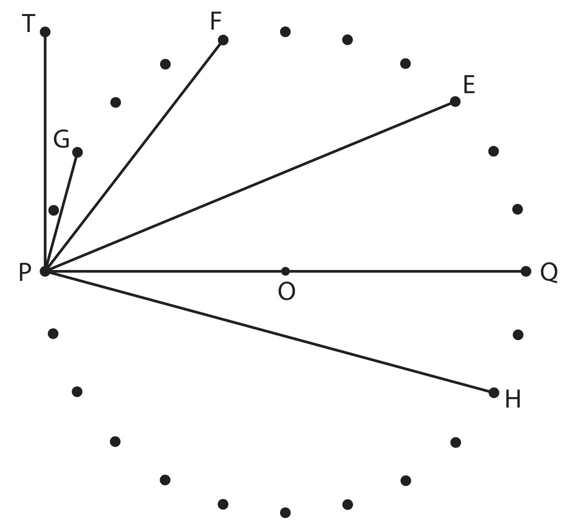
LAB 1.9

Name(s) ____________________________

Tangents and Inscribed Angles



1. In the figure at right, find the intercepted arc and the measure for each of the following angles.
   - **a.** $\angle QPA$ intercepts arc __________.
     $\angle QPA =$ __________
   - **b.** $\angle QPB$ intercepts arc __________.
     $\angle QPB =$ __________
   - **c.** $\angle QPC$ intercepts arc __________.
     $\angle QPC =$ __________
   - **d.** $\angle QPD$ intercepts arc __________.
     $\angle QPD =$ __________
   - **e.** Explain how you found the angle measures.



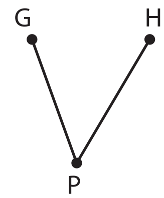
2. Segment *PT* is *tangent* to the circle (it touches it in exactly one point, *P*, which is called the *point of tangency*).
   - **a.** What arc is intercepted by $\angle QPT$?
     ____
   - **b.** What is the measure of $\angle QPT$?
     ____
3. **Important:** A segment that is tangent to a circle is ________________ to the radius at the point of tangency.
4. In the figure above, $\overline{PT}$ is tangent to the circle. Find the intercepted arc and the measure for each of the following angles. Explain on a separate sheet how you found the angle measures.
   - **a.** $\angle TPE$ intercepts arc __________.
     $\angle TPE =$ ____________
   - **b.** $\angle TPF$ intercepts arc __________.
     $\angle TPF =$ ____________
   - **c.** $\angle TPG$ intercepts arc __________.
     $\angle TPG =$ ____________
   - **d.** $\angle TPH$ intercepts arc __________.
     $\angle TPH =$ ____________
5. Use Problem 4 to check that the inscribed angle theorem still works if one side of the angle is tangent to the circle.

LAB 1.10
Soccer Angles

Name(s) ______________________

■ **Equipment:** Cardboard, unlined paper, Soccer Angles Worksheet, Soccer Circles Worksheet, scissors, straight pins

Soccer goals are 8 yards wide. Depending on where a player is standing, the angle she makes with the goalposts could be larger or smaller. We will call this angle the *shooting angle*. In this figure, the space between G and H is the goal, P is the player, and $\angle GPH$ is the shooting angle.



1. Place a piece of paper on a cardboard backing. Place two pins as goalposts two inches apart, centered, near the top of the page. Label the goalposts G and H.
2. Cut out these angles from the Soccer Angles Worksheet: 20°, 30°, 40°, 60°, 90°, 120°.
3. Using the paper angle and the pins, find the location of all points that are the vertices of a shooting angle equal to 40°.
4. Repeat Problem 3 with the other five angles cut from the Soccer Angles Worksheet, indicating clearly which points correspond to each shooting angle.

If you did this correctly, you should have found that the locations form arcs of circles that pass through G and H. You may remove the pins and ask your teacher for the Soccer Circles Worksheet, where the circles are drawn very accurately for you.

5. Label each arc with the measure of the corresponding shooting angle.
6. Which of the arcs is a half-circle? __________
7. The center of each of the six arcs is marked. Label each center so you will know which shooting angle it belongs to. (Use the notation C_{20}, C_{40}, and so on.)
8. Imagine a player is standing at C_{40}, the center of the 40° circle. What is the shooting angle there? ________________
9. Find the shooting angle for a person standing *at the center* of each of the remaining five circles. What is the pattern?

10. Find the center of the 45° circle without first finding points on the circle.

Name(s) ____________________

Discussion

A. The vertices of the 40° shooting angles all lie on a circle. They are the vertices of inscribed angles. Where is the arc that is intercepted by those angles?

B. How is the inscribed angle theorem helpful in understanding what happens in this activity?

For Questions C–E, refer to the Soccer Discussion Sheet.

C. Imagine a player is running in a direction parallel to the goal, for example, on line L_0. Where would he get the best (greatest) shooting angle? (Assume the player is practicing, and there are no other players on the field.)

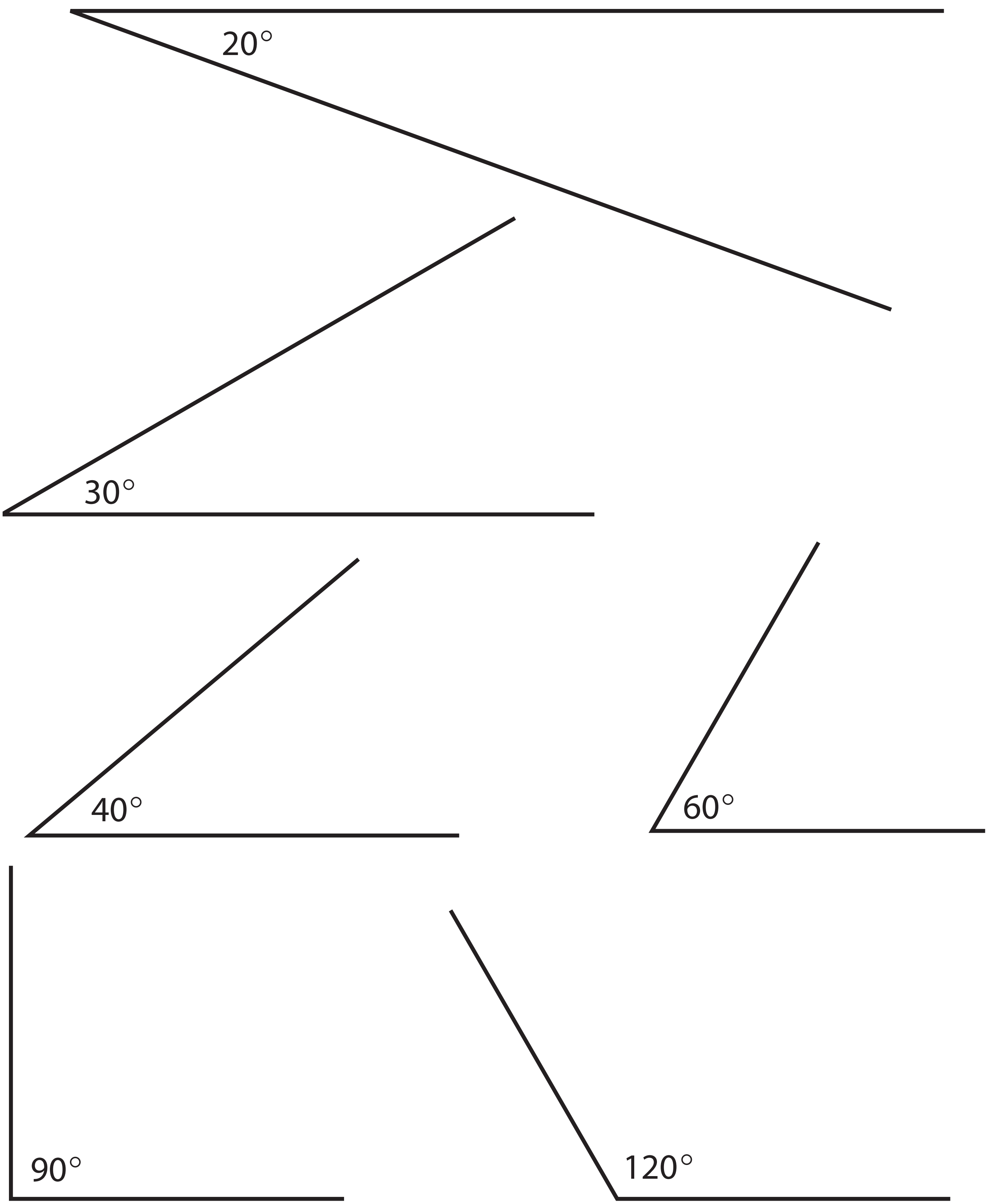
D. Imagine a player is running in a direction perpendicular to the goal, on a line that intersects the goal, for example, L_2. Where would she get the best shooting angle?

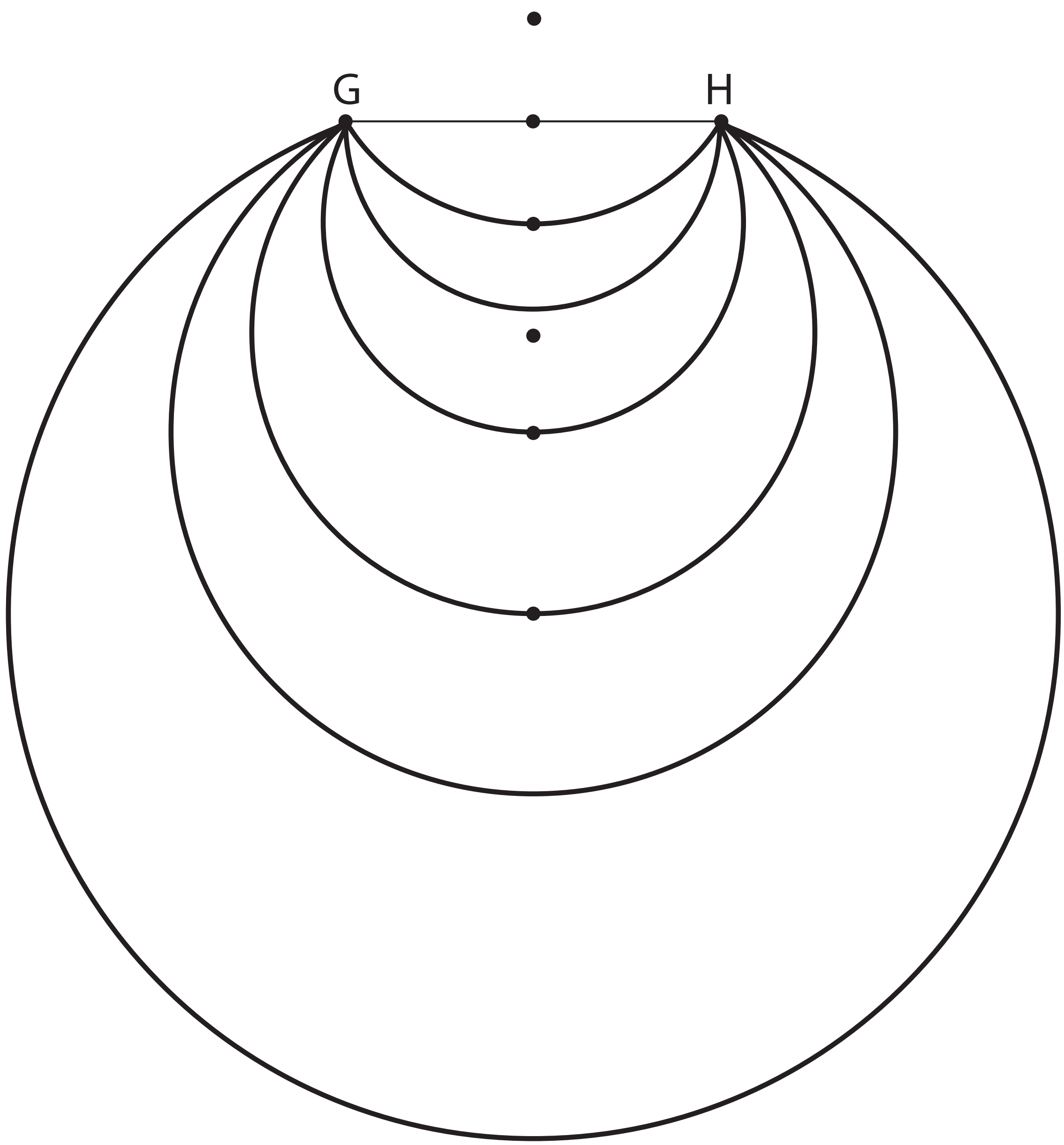
E. Imagine you are running in a direction perpendicular to the goal, on a line that does *not* intersect the goal, such as L_3 or L_4. Where would you get the best shooting angle?


G
H

LAB 1.10 Soccer Angles Worksheet

Soccer Angles (continued)




G
H

Soccer Angles (continued)