LESSON 14.7

Equations and Numbers

In this lesson we will discuss quadratic functions and equations in standard form, $y = a^2 + bx + c$ and $ax^2 + bx + c = 0$.

HOW MANY x-INTERCEPTS?

A quadratic equation may have 2, 1, or 0 real number solutions, depending on the number of x-intercepts on the graph of the corresponding function.

1. Sketch a parabola for whose equation:
   a. $a > 0$ and $c < 0$
   b. $a < 0$ and $c > 0$

2. Explain why a parabola for which a and c have opposite signs must intersect the x-axis.

3. Sketch a parabola to explain why if $a > 0$ and $V < 0$ there are two x-intercepts.

4. Fill the table with the number of x-intercepts for a quadratic function with the given signs for a and V. Justify each answer with a sketch.

	$V < 0$	$V = 0$	$V > 0$
$a > 0$	2	—	—
$a < 0$	—	—	—

(We do not consider the case $a = 0$, since then the function is no longer quadratic.)

5. How many x-intercepts are there if:
   a. $V = 0$?
   b. V and a have the same sign?
   c. V and a have opposite signs?

In Lesson 6 you found that $V = \frac{-b^2 + 4ac}{4a}$.

Definition: The quantity $b^2 - 4ac$, which appears under the radical in the quadratic formula, is called the *discriminant*, which is sometimes written Δ (the Greek letter *delta*).

6. Explain why we can write $V = -\Delta/(4a)$.

HOW MANY SOLUTIONS?

It turns out that the discriminant allows us to know the number of solutions of a quadratic equation. Refer to the table in problem 4 to answer the following questions.

7. If $\Delta = 0$, what is V? How many solutions are there?

8. If $\Delta > 0$,
   a. and $a > 0$, what is the sign of V? How many solutions are there?
   b. and $a < 0$, what is the sign of V? How many solutions are there?

9. If $\Delta < 0$,
   a. and $a > 0$, what is the sign of V? How many solutions are there?
   b. and $a < 0$, what is the sign of V? How many solutions are there?

The quadratic formula can be written:

$$\frac{-b \pm \sqrt{\Delta}}{2a}$$

10. **Summary** Using the quadratic formula, explain why,
   a. if $\Delta = 0$ there is only one solution;
   b. if $\Delta < 0$ there are no real solutions;
   c. if $\Delta > 0$ there are two real solutions.

11. Explain why if a and c have opposite signs, the discriminant cannot be negative.

SUM AND PRODUCT OF THE SOLUTIONS

12. In the case where there are two solutions $x_1 = \frac{-b + \sqrt{\Delta}}{2a}$ and $x_2 = \frac{-b - \sqrt{\Delta}}{2a}$,

a. what is $x_1 + x_2$?

b. what is the average of x_1 and x_2? (How is this related to the axis of symmetry?)

c. what is $x_1 \cdot x_2$?

The sum of the solutions of a quadratic equation is $S = -b/a$, and the product is $P = c/a$. This provides a quick way to check the correctness of the solutions to a quadratic.

Example: Phred solved the quadratic equation $2x^2 + 5x - 8 = 0$ and got $\frac{-5 \pm \sqrt{89}}{2}$. To check the correctness of the answer, he added the two roots, hoping to get $S = -b/a = -5/2$. Conveniently, the $\sqrt{89}$ disappeared:

$$\frac{-5 + \sqrt{89}}{2} + \frac{-5 - \sqrt{89}}{2} = \frac{-10}{2}$$

Since $-10/2 \neq -5/2$, Phred must have made a mistake.

Solve, and check the correctness of your answers, with the help of S and P (or by substituting in the original equation).

13. $2x^2 + 5x - 8 = 0$

14. $2x^2 - 8x + 5 = 0$

15. $-8x^2 + 3x + 5 = 0$

16. $-2x^2 - 5x - 1 = 0$

KINDS OF NUMBERS

The first numbers people used were whole numbers. It took many centuries to discover more and more types of numbers. The discovery of new kinds of numbers is related to the attempt to solve more and more equations. The following equations are examples.

a. $x + 2 = 9$ b. $x + 9 = 2$

c. $2x = 6$ d. $6x = 2$

e. $x^2 = 9$ f. $x^2 = 10$

g. $x^2 = -9$

17. Pretend you know about only the *natural numbers*. (These are the positive whole numbers.) List the equations a-f that can be solved.

18. Pretend you know about only the *integers*. (These are positive and negative whole numbers and zero.) List the equations a-f that can be solved. Find one that has two solutions.

19. Pretend you know about only the *rational numbers*. (These are all fractions, positive, negative, and zero. Of course, integers are included, since for example 3 = 6/2.) List the equations a-f that can be solved.

20. The *real numbers* include all rational and irrational numbers. Which equations can be solved if you know about all the real numbers?

Natural numbers, integers, rational numbers, and real numbers can all be found on a one-dimensional number line. However, to solve equation (g), you need to get off the number line. The solution is a *complex number*, and it is written $3i$. The number i is a number one unit away from 0, but off the number line. It is defined as a number whose square is -1:

$$i^2 = -1.$$

Complex numbers cannot be shown on a line. They require a two-dimensional number plane. You will learn more about them in future math classes.

21. Create an equation whose solution is

a. a natural number;

b. an integer, but not a natural number;

c. a rational number, but not an integer;

d. an irrational number.

22. Create an equation that has no real number solution.

23. Research Find out about complex numbers.