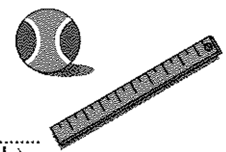
LESSON 11.1

Sums of Geometric Sequences

You will need:

a ball

yardstick (or meterstick)



THE BOUNCING BALL

When you drop a ball, it bounces back, but not quite to the height from which you dropped it.

1. Do an experiment in which you drop a ball from various heights and see to what height it bounces back. Use a yardstick or meterstick to make your measurements. Make a table like this.

Dropped from	Bounced to	Ratio	Difference
—	—	—	—

2. As you vary the height, what remains closer to constant, the ratio or the difference?

For a certain "ideal" ball, the bounce-height to drop-height ratio (or *bounce ratio*) is consistently 0.8. The ball is dropped from a height of two meters.

3. a. How high does it bounce on the first, second, and third bounces?

b. How many bounces until it bounces to fewer than 80 centimeters?

c. How many bounces until it bounces to fewer than 10 cm?

4. What is the total distance traveled by the ball (both down and up) if someone catches it at the top of its bounce after:

a. 2 bounces? b. 20 bounces?

5. Make a guess about the total distance traveled by the ball after 200 bounces. Justify your guess.

USING SYMBOLIC NOTATION

Say the bounce ratio is r. Then we have:

$$\frac{\textit{bounce height}}{\textit{drop height}} = r$$

Or: $\textit{bounce height} = r \cdot \textit{drop height}$

Assume that the initial drop height is H.

6. How high does the ball bounce on the first, second, third, and fourth bounces? Express your answers in terms of H and r.

To analyze the problem of the total distance traveled, it is easier to separate the upwards and downwards motions. First find the downwards distance traveled in the first four bounces.

$$D_4 = H + Hr + Hr^2 + Hr^3$$

As you see, the terms of the sum form a *geometric sequence* having first term H and *common ratio* r.

7. Write an expression for D_6 the downwards distance traveled in the first six bounces.

8. What is the last exponent in the expression for the downwards distance traveled in the first n bounces? Explain why the exponent is not the same as the number of bounces.

9. Write an expression for the upwards distance traveled in:

a. the first four bounces, U_4;

b. the first six bounces, U_6.

10. What is the last exponent in the expression for the upwards distance traveled in the first n bounces? Why does this differ from the expression for the downwards distance?

FINDING THE SUM

Here is a shortcut for calculating the sum of a geometric sequence. We will use the example of the ideal ball having bounce ratio 0.8, dropped from a height of two meters, and caught at the top of its fourth bounce. First write the downwards motion.

Eq. 1: $D_4 = 2 + 2(0.8) + 2(0.8)^2 + 2(0.8)^3$

Do not calculate the sum! You will soon see why.

Multiplying both sides by 0.8, we get:

Eq. 2: $D_4(0.8) = 2(0.8) + 2(0.8)^2 + 2(0.8)^3 + 2(0.8)^4$

Subtracting one equation from the other:

Eq. 1-Eq. 2: $D_4 - D_4 \cdot (0.8) = 2 - 2(0.8)^4$

11. Explain why there are so few terms after subtracting.

12. *Solve* for D_4. (Hint: Factor, then divide.)

13. Use this *multiply-subtract-solve* technique to find U_4. You found an expression for U_4 in problem 9.

14. What is the total distance traveled by the ball in four bounces?

When adding only four terms, the multiply-subtract-solve technique is not much of a shortcut. However, when adding large numbers of terms, it is extremely convenient. For example, for 20 bounces, you would start by writing:

$D_{20} = 2 + 2(0.8) + ... + 2(0.8)^{18} + 2(0.8)^{19}$

15. Explain why in this case the last terms do not contribute very much to the sum.

16. Use the multiply-subtract-solve technique to check the correctness of your answers for problems 4b and 5.

OTHER BOUNCE RATIOS

17. What is the total distance traveled in 200 bounces by a ball having the following bounce ratios, after being dropped from a height of two meters?

a. a super-ball, having bounce ratio 0.9

b. a flat basketball, having bounce ratio 0.3

18. Repeat problems 3-5 for a real ball. (First, you must find the bounce ratio, perhaps by averaging the ratios you found in problem 1.) Verify your predictions for problem 3 with experiments.

An absent-minded professor invents a hyper-ball having a bounce ratio of 1.1.

19. Repeat problems 3-5 for the hyper-ball.

20. Repeat problems 3-5 for a defective hyper-ball having a bounce ratio of only 1.

21. Summary Summarize what you learned about the sum of geometric sequences.

a. Explain the multiply-subtract-solve method. (What does one multiply by? What does one subtract? What does one solve for, and how?)

b. What is the effect of the common ratio on the sum? (What if r is less than 1? What if it is equal to 1? What if it is greater than 1?)

22. Generalization Use the multiply-subtract-solve technique for each sum S.

a. $S = a + ar + ar^2 + ... + ar^{n-1}$

b. $S = a + ar + ar^2 + ... + ar^n$

DISCOVERY ***FOUR NUMBERS***

23. a. Replace each box with one of the numbers: 1, 2, 3, 4. (Use each number exactly once.)

$$\frac{\square}{\square} + \frac{\square}{\square}$$

How many possible arrangements are there?

b. Which arrangement gives the smallest sum? What is the smallest sum?

c. Which arrangement gives the largest sum? What is the largest sum?

d. Are the arrangements that give the smallest and the largest answer *unique*? That is, is there only one arrangement that gives the same sum?

24. Repeat problem 23 for $\frac{\square}{\square} - \frac{\square}{\square}$, this time finding the arrangements that give the smallest and the largest difference. How are the smallest and the largest difference related? Explain.

25. Repeat problem 23 for $\frac{\square}{\square} \cdot \frac{\square}{\square}$, this time finding the arrangements that give the smallest and the largest product. How are the smallest and the largest product related? Explain.

26. Repeat problem 23 for $\frac{\square}{\square} \div \frac{\square}{\square}$, this time finding the arrangements that give the smallest and the largest quotient. How are the smallest and the largest quotient related? Explain.

27. Choose four numbers a, b, c, d such that $a < b < c < d$. Repeat problems 23-26 for these numbers. Compare your answers with other students' answers. Were you able to use the answers from problems 23-26 to help you?

28. **Report** Write a report summarizing your findings in problems 23 through 27. Describe the strategies you used for finding the smallest and the largest values. Explain why you were sure that they were the smallest and the largest.