LESSON 8.5

Ideal Population Growth

MATHEMATICAL MODELS

Exponents are useful for making mathematical descriptions of many kinds of growth, including population growth and spread of infectious disease. A mathematical description, or *mathematical model,* usually involves simplifying the real-world situation. Even though some of the idealized situations you study in this course may seem unrealistic, they will help you learn techniques that can be applied to more complicated real-world data.

Bacterial growth is one such situation. In research laboratories, bacteria used for biological studies are grown under controlled conditions. Although no real populations would grow as predictably as the ones described in this chapter, bacterial populations over short periods of time do approximate this kind of growth.

A DOUBLING POPULATION

1. A colony of bacteria is being grown in a laboratory. It contains a single bacterium at 12:00 noon (*time 0*), and the population is doubling every hour. How long do you think it would take for the population to exceed 1 million? 2 million? Write down your *guesses* and compare them with other students' guesses.

2. Make a table of values showing how the population in problem 1 changes as a function of time. Find the population one hour from now, two hours from now, etc. Extend your table until you can answer the questions asked in problem 1. How close were your guesses?

3. Add a third column to your table, writing the population each time as a power of 2.

4. What would the population be after x hours? (Write this as a power of 2.)

HOW MUCH MORE THAN? HOW MANY TIMES AS MUCH?

To determine the rate at which the population is increasing, we compare the populations at different times.

5. Compare the population after 8 hours with the population after 5 hours.
   a. *How much more* is the population after 8 hours? (Compare by subtracting.)
   b. *How many times* as much is it? (Compare by dividing.)
   c. Which of your answers in (a) and (b) can be written as a power of 2? What power of 2 is it?

6. Repeat problem 5, comparing the population after 7 hours with the population after 3 hours.

7. One of the questions, *How much more than?* or, *How many times as much?* can be answered easily with the help of powers of 2. Which question? Explain.

8. Make the comparisons below, answering the question: How many times as much? Write your answers as powers of 2.
   a. Compare the population after 12 hours with the population after 10 hours.
   b. Compare the population after 9 hours with the population after 4 hours.
   c. Compare the population after 4 hours with the population after 12 hours.

9. Compare each pair of numbers. The larger number is how many times as much as the smaller number? Write your answer as a power. In (d), assume x is positive.
   a. 2^6 and 2^9 b. 2^9 and 2^{14}
   c. 2^{14} and 2^6 d. 2^x and 2^{x+3}

A TRIPLING POPULATION

A colony of bacteria being grown in a laboratory contains a single bacterium at 12:00 noon (time 0). This population is *tripling* every hour.

10. Make the comparisons below, answering the question: How many times as much? Write your answers as powers of 3. (Hint: It may help to start by making a table showing how the population changes as a function of time.)

a. Compare the population after 12 hours with the population after 10 hours.

b. Compare the population after 9 hours with the population after 4 hours.

c. Compare the population after 4 hours with the population after 12 hours.

11. Compare each pair of numbers. How many times as much as the smaller number is the larger? Write your answer as a power. In (d), assume x is positive.

a. 3^6 and 3^9 b. 3^9 and 3^{14}

c. 3^{14} and 3^6 d. 3^x and 3^{x+5}

12. By what number would you have to multiply the first power to get the second power? Write your answer as a power.

a. $3^5 \cdot \_\_ = 3^{15}$ b. $3^8 \cdot \_\_ = 3^{15}$

c. $3^{11} \cdot \_\_ = 3^{15}$ d. $3^0 \cdot \_\_ = 3^{15}$

MULTIPLYING AND DIVIDING POWERS

In a power, the exponent tells how many times the base has been used as a factor. For example, 4^2 means $4 \cdot 4$, and 4^3 means $4 \cdot 4 \cdot 4$, therefore:

$$4^2 \cdot 4^3 = (4 \cdot 4) \cdot (4 \cdot 4 \cdot 4) = 4^5.$$

Use this idea to multiply powers.

13. Write the product as a power of 3.

a. $3^7 \cdot 3^3 =$ b. $3^5 \cdot 3^5 =$

c. $3^8 \cdot 3^2 =$ d. $3^8 \cdot 3^0 =$

14. Write the product as a power of 5.

a. $5^4 \cdot 5^3 =$ b. $5^4 \cdot 5^6 =$

c. $5^4 \cdot 5^9 =$ d. $5^0 \cdot 5^0 =$

When you divide, the quotient tells you how many times as much the numerator is than the denominator. For example, $4^5/4^2$ means *what times 4^2 equals 4^5?* Since $4^2 \cdot 4^3 = 4^5$, you have $4^5/4^2 = 4^3$. Use this idea to divide powers.

15. Write the quotient as a power of 2.

a. $\frac{2^{11}}{2^6}$ b. $\frac{2^6}{2^3}$

c. $\frac{2^{11}}{2^3}$ d. $\frac{2^{11}}{2^0}$

16. Write the quotient as a power of 3.

a. $\frac{3^7}{3^5}$ b. $\frac{3^6}{3^4}$

c. $\frac{3^{x+2}}{3^x}$ d. $\frac{3^{11}}{3^0}$

17. Use what you have learned in this lesson to find x.

a. $5^x \cdot 5^3 = 5^9$

b. $2^3 \cdot x^4 = 2^7$

c. $\frac{8^{66}}{8^x} = 8^{54}$

d. $\frac{10^{x+3}}{10^4} = 10^{a+1}$

18. Summary

a. Describe the patterns you found in multiplying and dividing powers.

b. Give examples to show how patterns can make it easier to multiply and divide powers.

c. In each multiplication and division problem, 15-17, *the bases of the powers are the same.* Does the pattern you described in (a) work if the bases are not the same? Explain, using examples.

19. Generalization Use the patterns you found in this lesson to rewrite each expression as a single power.

a. $5^x \cdot 5^y$ b. $a^x \cdot a^y$

c. $\frac{3^y}{3^x}$ d. $\frac{6^{x+5}}{6^x}$

e. $6^x \cdot 6^x$ f. $6^0 \cdot 6^x$