CHAPTER 5

The colorful spiral of a lollipop

Coming in this chapter:

Exploration Build as many rectangles as you can with one x^2, ten x-blocks, and any number of yellow blocks.

Build as many rectangles as you can with x^2, 18, and any number of x-blocks.

Sums and Products

LESSON 5.1

Constant Sums

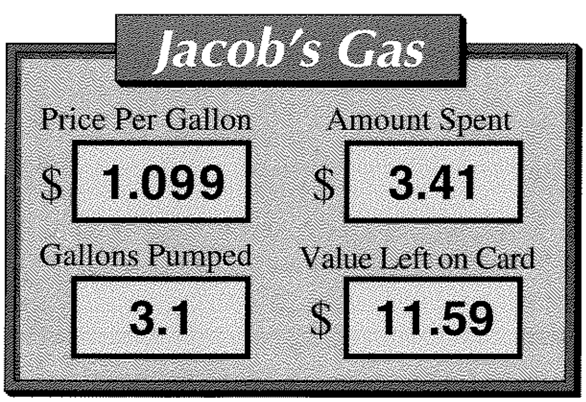
You will need: graph paper

AT THE GAS STATION

When Oliver and Alice pulled up to the self-serve island at Jacob's gas station, they noticed a new sign:

Buy Gas Card in Office

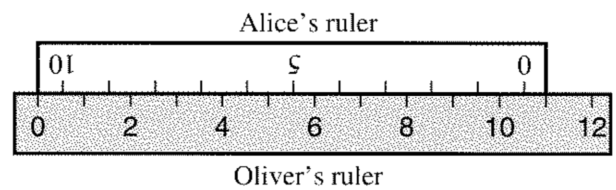
They went into the office, which was decorated with photographs and cartoons. The attendant Harold explained to them that he could sell them a gas card for any amount from 5 to 100 dollars. They would put it in the special slot in the pump, and pump gas as usual. The value of the card would automatically go down, and a display on the pump would indicate the value left in the card. After getting gas, there would be no need to go back to the office, unless they wanted to trade the card back for cash. (This could be done only if the card had less than $5 left on it.) Or they could use the remaining money left on the card the next time they stopped at Jacob's.



1. Look at the dials in the figure. How much did Oliver and Alice pay for their gas card?
2. Oliver and Alice plan to buy about $10.00 worth of gas. List at least five other pairs of numbers that will appear on the last two dials while they are pumping gas.
3. When exactly 11 gallons have been pumped, what numbers will appear on the four dials?
4. **Generalization** When D dollars have been spent, what is the value left on the card?
5. When G gallons have been pumped,
   a. how many dollars have been spent?
   b. what is the value left on the card?

FUNCTION DIAGRAMS FROM RULERS

Alice wanted to know how long her ruler was. Oliver suggested she measure it with a longer ruler, as in this figure.



6. How long is her ruler?

Oliver had to write about function diagrams for algebra. (His class was using this textbook, and in a curious coincidence, they were doing exactly this page!) He decided to use the rulers as a way to get tables of x- and y-values and build a function diagram from them. He used the rulers setup to create a table that started this way.

x	y
1	9.5
2	8.5
3	7.5

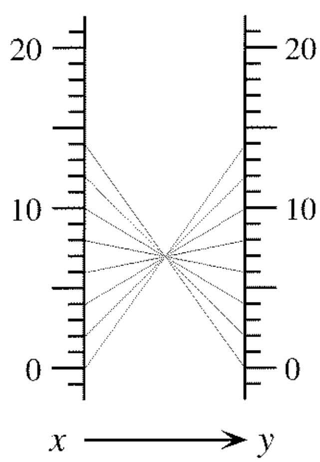
7. Describe the pattern for the numbers in the table. Does it matter which ruler you use for x and which for y? Explain.

8. Write a function of the type $y =$ *an expression in terms of* x for Oliver's table.

9. Make a function diagram for Oliver's table. (Use at least five in-out lines.)

10. Use rulers to create two more tables, and for each, write a function and make a function diagram. At least one of them should match 0 with a number other than a whole number.

11. How could you set up rulers to get this function diagram? Explain.



All the function diagrams you just drew have something in common. For each one, the sum of all the (x, y) pairs is a constant. We could call them *constant sum* functions.

12. Summary Write an illustrated summary describing what you noticed about diagrams of constant sum functions. It should include, but not be limited to, examples and answers to the following questions:

- Do the in-out lines meet in one point?
- If they do, could you predict the position of this point if you knew the value of the constant sum?

GRAPHS OF CONSTANT SUMS

13. a. On a pair of axes, plot these (x, y) pairs.
$(2, 4)$ $(4, 2)$ $(-1, 7)$ $(8, -2)$

b. In words, we could describe the pattern of the (x, y) pairs by saying that the sum of x and y is always six. How would you write this using algebra?

c. Find three more (x, y) pairs that fit this pattern, and add the points to your graph.

d. Connect all the points with a line or curve. Describe the graph.

14. a. Find points such that $x + y < 6$. Where are they in relation to the graph in problem 13?

b. Repeat for $x + y = 6$.

c. Repeat for $x + y > 6$.

15. Find a point (x, y) such that $x = y$ and $x + y = 6$. Label it on the graph.

16. Choose a positive value for S and make a table of (x, y) pairs that satisfy the equation $x + y = S$. Use your table to make a graph.

17. Experiment with some other constant sum graphs. Try several different positive values for S. For each one, make a table of at least five (x, y) pairs having the sum S. Then draw a graph. Draw all your graphs on the same pair of axes.

18. Do any of the lines go through the origin? If not, do you think you could pick a number for your sum so that the line would go through the origin? Explain.

19. Repeat your investigations for equations of the form $x + y = S$, where S is negative. Keep a record of what you try, using tables and graphs.

20. Report Write an illustrated report summarizing your findings about constant sum graphs. Your report should include neatly labeled graphs with accompanying explanations. Include answers to the following questions:

- Were the graphs straight lines or curved, or were there some of each?
- Without drawing the graph, could you now predict which quadrants the graph would be in, if you knew the value of S? Explain.
- Without drawing the graph, could you predict the x-intercepts and y-intercepts of the graph, if you knew the value of S? Explain.
- What determines whether the graph slopes up or down as it goes from left to right? Could you predict this without graphing if you knew the value of S? Explain.
- Do any of your graphs intersect each other? If so, which ones? If not, why not?

LESSON 5.2

Constant Products

MILES PER GALLON

If you plan to take a trip of 100 miles, the amount of gas you need depends on how many miles per gallon your vehicle gets. Some very large recreational vehicles get only about 5 miles per gallon, while a scooter can get 100 miles per gallon.

1. Copy and complete the table to show how many gallons of gasoline you should buy if your vehicle gets the mileage indicated. Continue the table up to 100 miles per gallon.

Mileage (miles per gallon)	Gasoline needed (gallons)	Total trip distance (miles)
5	——	100
10.5	——	100
20	——	100

2. Graph the (x, y) pairs in the first two columns of the table.

3. Describe your graph in words. If you were to extend your graph, would it go through the origin? Would it touch or cross the axes? Explain.

CONNECTING THE DOTS

4. Make a table containing these points and plot the (x, y) pairs on a Cartesian graph.

$(2, 12)$ $(3, 8)$ $(4, 6)$ $(8, 3)$

5. Describe the pattern of the (x, y) pairs in problem 4

a. in words; b. using algebra.

6. a. Find five more (x, y) pairs that fit this pattern and add the points to your table and graph. Use positive values for x. Include some fractional values.

b. Add five more (x, y) pairs to your table and graph. This time use negative values for x, including some fractional values.

7. Study the points on your graph. If necessary, add more points so that you can answer the following questions.

a. Which quadrants do your points lie in? Why?

b. Can you find a point on the y-axis that fits the pattern? Can you find a point on the x-axis? Explain.

c. If you were to connect the points with a smooth curve, would the curve go through the origin? Explain.

8. Add to your graph a point that fits the pattern and

a. has an x-value less than 1/2;

b. has a y-value less than 1/2;

c. has an x-value greater than 24;

d. has a y-value less than -24.

9. Study your answers to problems 4-8. Then very carefully connect the points with a curve. Your curve should have two parts *that are not connected* to one another.

a. Describe the graph.

b. Explain why the two parts are not connected.

10. For (a-d), find several pairs of numbers (x, y) that satisfy the description. Plot these points on your graph.

a. x is positive and xy is more than 24.

b. x is positive and xy is less than 24.

c. x is negative and xy is more than 24.

d. x is negative and xy is less than 24.

11. Plot a point (x, y) such that $xy = 24$ and $x = y$.

We could call the curve you drew in problem 9 a *constant product* graph, since the product of the coordinates of every point is the same number. We could graph many other *constant product* graphs of the form $xy = P$, where P could be any number we choose.

12. Experiment with the graphs of some equations of the form $xy = P$. Try several different positive values for P. Then try several different negative values for P. For each one, make a table of at least eight (x, y) pairs having the same product. Then draw a graph. Draw all your graphs on the same pair of axes.

13. Report Write a report summarizing your findings about constant product graphs. Your report should include neatly labeled graphs with accompanying explanations. Include answers to the following questions:

- What is the shape of the graph?
- Without drawing the graph, could you now predict which quadrants the graph would be in, if you knew the value of P? Explain.
- Do any of the graphs go through the origin? If not, do you think you could find a value of P so that the graph would go through the origin? Explain.
- Where can you find points whose product is not P?
- Comment on anything you notice about the x-intercepts and y-intercepts.
- Do any of your graphs intersect? Explain why or why not.

OTHER GRAPHS

In order to graph some functions, Tomas made tables of values, plotted the points, and connected the dots. (For one of the equations, he tried two different ways.) He asked his teacher if he had done it right. Mr. Stephens answered that the individual points had been plotted correctly, but he asked Tomas to think about how he had connected them. He said, *"Every point on the graph, even the ones obtained by connecting the dots, must satisfy the equation."* Tomas didn't understand. Mr. Stephens added, "Check whether you connected the dots correctly, by substituting a few more values of x into the equation. Use your calculator to see if the y-value you get is on the graph you drew." Tomas still didn't understand.

Name: Tomas

$y = x - 2$

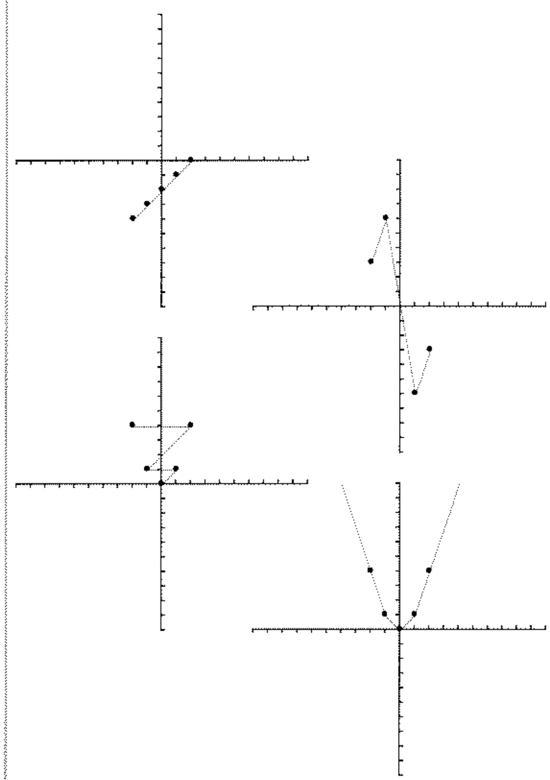
x	y
-2	-4
-1	-3
0	-2
1	-1
2	0

$y = x^2$

x	y
-2	4
-1	1
0	0
1	1
2	4

$y = -6/x$

x	y
-2	3
-1	6
0	imp.
1	-6
2	-3

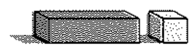


14. **Report** Explain how Tomas can improve his graphs. Show your calculations. Give Tomas advice he can understand, on:

- how to label axes, points, and graphs;
- how to connect the dots correctly;
- how to extend the graph to the left and right;
- how a calculator can help.

LESSON 5.3 The Distributive Law

You will need:

the Lab Gear 

HOW MANY TERMS?

For each multiplication, write an equation of the form *length · width equals area.* (You may use the Lab Gear and the corner piece to model the multiplication by making a rectangle.) In your expression for the area, combine like terms.

1. $x(2x + 5)$
2. $2x(y - 2)$
3. $y(2y + 2 - x)$
4. $(2x + 2)(3x - 5)$
5. $(x + 2)(3y + 1)$
6. $(x + 2)(y - 3x + 1)$

For each multiplication, write an equation of the form *length · width · height equals volume.* (You may want to use the Lab Gear and the corner piece to model the multiplication by making a box.) In your expression for the volume, combine like terms.

7. $x(x + 2)(x + 5)$
8. $y(x + 2)(y + 1)$
9. $x(x + 5)(x + y + 1)$

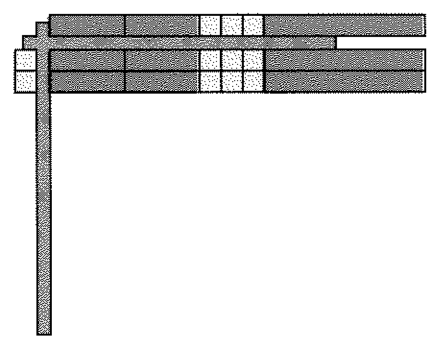
Definitions: A polynomial having two terms is called a *binomial;* one having three terms is called a *trinomial.* A polynomial having one term is called a *monomial.*

10. **Report** In problems 1-9, you multiplied two or three polynomials of degree 1. In each case, the product was also a polynomial. Write a report describing the patterns you saw in the products. You should use the words *monomial, binomial,* and *trinomial.* Give examples and illustrate your work with drawings of the Lab Gear. Your report should address the points listed below, but should also include any other observations you made.
    - What determines the degree of the product?
    - What determines the number of terms in the product?
    - Compare problems having one variable to problems having two variables.

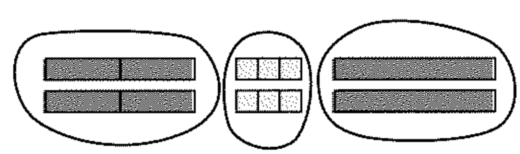
DIVISION AND THE DISTRIBUTIVE LAW

As you probably remember, you can use the corner piece to model division.

Example: Simplify $\frac{4x + 6 + 2y}{2}$



In some cases, you can use the Lab Gear in another way to show that a division like this one can be thought of as three divisions.



11. What is the result of the division?

Simplify these expressions, using the Lab Gear if you wish.

12. $\frac{10x + 5y + 15}{5}$
13. $\frac{2x + 4}{x + 2}$

14. $\frac{x^2 + 4x + 4}{x + 2}$

15. $\frac{3(y - x) + 6(x - 2)}{3}$

Another way to simplify some fractions is to rewrite the division into a multiplication and use the distributive law.

Example: To simplify $\frac{6x + 4 + 2y}{2}$:

- Rewrite the problem as a multiplication.

$$\frac{1}{2}(6x + 4 + 2y)$$

- Apply the distributive law.

$$\frac{1}{2} \cdot 6x + \frac{1}{2} \cdot 4 + \frac{1}{2} \cdot 2y$$

- Simplify.

$$3x + 2 + y$$

You can see that we could have divided every term in the numerator by 2. That is:

$$\frac{6x + 4 + 2y}{2} = \frac{6x}{2} + \frac{4}{2} + \frac{2y}{2}.$$

The single division problem was equivalent to three divisions. This example illustrates *the distributive law of division over addition and subtraction.*

Divide.

16. $\frac{9x + 6y + 6}{3}$

17. $\frac{3x^2 + 2x}{2x}$

18. $\frac{6x^2 + 4x}{2x}$

19. $\frac{2(x + 3) + 5(x + 3)}{x + 3}$

DISTRIBUTIVE LAW PRACTICE

Find these products, using the Lab Gear or any other method.

20. $2x(x - 1)$

21. $y(y + 4)$

22. $3x(x + y - 5)$

23. $(x + 5)(3x - 2)$

24. $(2x + 4)(x + y + 2)$

25. $(2y - x - 3)(y + x)$

Write equivalent expressions without the parentheses. Combine like terms.

26. $z(x + y) + z(x - y)$

27. $z(x + y) + z(x + y)$

28. $z(x + y) + x(z + y)$

29. $z(x + y) - x(z + y)$

MULTIPLYING BINOMIALS

The following problems involve multiplying two binomials of the form $ax + b$ or $ax - b$. Multiplications like this arise often in math. As you do them, look for patterns and shortcuts.

30. $(3x + 2)(5x + 6)$

31. $(3x - 2)(5x + 6)$

32. $(3x + 2)(5x - 6)$

33. $(ax + 2)(3x + d)$

34. $(2x + b)(cx - 3)$

35. When you multiply two binomials of the form $ax + b$ or $ax - b$,
   a. what is the degree of the product?
   b. how many terms are in the product?

36. When multiplying two binomials of the form $ax + b$ or $ax - b$, how do you find
   a. the coefficient of x^2?
   b. the coefficient of x?
   c. the constant term?

LESSON 5.4

Factoring Trinomials

You will need:

the Lab Gear 

LAB GEAR RECTANGLES

1. **Exploration**

   a. Use the Lab Gear to make as many different rectangles as you can with one x^2-block, ten x-blocks, and any number of yellow blocks. For each one, write a multiplication equation to show that *area = length times width.* Look for patterns.

   b. Use the Lab Gear to make as many different rectangles as you can with one x^2-block, 18 yellow blocks, and any number of x-blocks. For each one, write a multiplication equation to show that *area = length times width.* Look for patterns.

2. Use the Lab Gear to help you find the other side of the rectangle having the given area. Look for patterns. One is impossible.

Side	Area
a. $x + 4$	$x^2 + 9x + 20$
b. $x + 3$	$x^2 + 4x + 3$
c. $x + 6$	$x^2 + 6x + 8$
d. $x + 1$	$x^2 + 3x + 2$
e. $x + 4$	$x^2 + 7x + 12$

FACTORS AND PRODUCTS

Definition: To *factor* means to write as a product.

For example, two ways of factoring 12 are to write it as $6 \cdot 2$ or as $4 \cdot 3$. Some polynomials can be factored. With the Lab Gear we model this by making a rectangle or a box.

3. By making a Lab Gear rectangle and writing a related multiplication equation, show that $5y + y^2$ can be written as the product of a monomial and a binomial.

You have factored the polynomial $5y + y^2$.

4. By making a rectangle with the Lab Gear and writing a related multiplication equation, show that the trinomial $x^2 + 3x + 2$ can be written as a product of two binomials.

As this problem showed, some trinomials of the form $x^2 + bx + c$ can be factored.

5. Factor each trinomial into the product of two binomials. It may help to use the Lab Gear to make rectangles.

   a. $x^2 + 8x + 7$

   b. $x^2 + 8x + 12$

   c. $x^2 + 8x + 15$

6. Are there any more trinomials of the form $x^2 + 8x +$ ___ that can be factored into two binomials? If so, write and factor them. If not, explain.

7. Factor each trinomial into the product of two binomials. It may help to use the Lab Gear to make rectangles.

   a. $x^2 + 13x + 12$

   b. $x^2 + 8x + 12$

   c. $x^2 + 7x + 12$

8. Are there any more trinomials of the form $x^2 +$ __$x + 12$ that can be factored into two binomials? If so, write and factor them. If not, explain.

THE THIRD DEGREE

9. Factor these third-degree polynomials into a product of three first-degree polynomials. Making a box with the Lab Gear may help.

a. $x^2y + 5xy + 6y$
b. $x^3 + 5x^2 + 6x$
c. $y^3 + 5y^2 + 6y$
d. $xy^2 + 5xy + 6x$

10. Describe a strategy to factor the polynomials above without the Lab Gear.

11. Factor, using the Lab Gear if you need to, $x^2y + x^2 + 5xy + 5x + 6y + 6$.

PLUS AND MINUS

12. a. Use the corner piece and the Lab Gear to show the multiplication

$$(y + 4)(y + 3).$$

Write the product.

b. How many blocks of each type were needed to show the product?

13. a. Use the corner piece and the Lab Gear to show the multiplication

$$(y - 4)(y + 3).$$

Write the product.

b. Compare the number of blocks of each type used to show this product with the number of blocks used in problem 12.

14. Write another multiplication that requires one y^2-block, seven y-blocks, and twelve 1-blocks to show the product. Model it with the blocks and write the product. Compare work with your classmates. Is there more than one possibility?

MISSING TERMS

Supply the missing terms. Then compare your answers with your classmates' answers.

15. $x^2 + 15x +$ __ $= (x +$ __$)(x +$ __$)$

16. $x^2 - 7x +$ __ $= (x -$ __$)(x -$ __$)$

17. $x^2 +$ ___$x + 15 = (x +$ __$)(x +$ __$)$

18. $x^2 -$ ___$x + 7 = (x -$ __$)(x -$ __$)$

19. Which problems, 15-18, have more than one answer? Explain.

FACTORING BY TRIAL AND ERROR

20. If possible, factor each trinomial into a product of binomials. Try to do it without using the Lab Gear.

a. $x^2 + 5x + 6$
b. $a^2 + 11a + 30$
c. $m^2 + 20m + 100$
d. $p^2 + 2p + 1$

21. Factor.

a. $x^2 - 5x + 6$
b. $x^2 - 13x + 12$
c. $x^2 - 8x + 15$
d. $x^2 - 9$

22. Factor.

a. $6x^2 + 5x + 1$
c. $6x^2 + x - 1$
d. $6x^2 - x - 1$

23. Factor.

a. $x^4 - 8x^2 + 15$
b. $x^4 - 8x^2 + 16$

WHAT'S YOUR PROBLEM?

24. Make up six trinomials of the form $x^2 + bx + c$. Four should be factorable, and two should be impossible to factor. Exchange with another student, and try to factor each other's trinomials.

THINKING WRITING

5.A Analyzing Graphs

You will need:

graph paper

CONSTANT PRODUCTS

1. a. On the same pair of axes, graph the constant product function $xy = 24$ and the constant sum function $x + y = 10$.

b. Find and label the points where these two graphs intersect.

c. Add the graph of $x + y = 4$ to the same pair of axes. Does it intersect either graph?

2. If possible, factor each trinomial.

a. $x^2 + 10x + 24$

b. $x^2 + 4x + 24$

3. Explain the relationship between problem 1 and problem 2.

4. Make a large graph of the constant product equation. $xy = 36$. Show both branches on your graph.

5. On the graph of $xy = 36$, find two (x, y) pairs whose sum is 13. Plot and label these points, and connect them with a straight line. What is the equation of the line connecting these two points?

6. Add to your graph several lines of the form $x + y = S$, where S is an integer, as described below. Draw at least three lines

a. that intersect the graph of $xy = 36$ in the first quadrant. (Label the graphs and the points of intersection.)

b. that intersect the graph of $xy = 36$ in the third quadrant. (Label the graphs and the points of intersection.)

c. that never intersect the graph of $xy = 36$.

7. Consider the expression $x^2 +$ ___ $x + 36$. What numbers could you put in the blank to get a trinomial that can be factored? Explain your answer, giving examples.

CONSTANT SUMS

8. Make a large graph of the constant sum $x + y = 12$.

9. a. Find many (x, y) pairs whose product is 20.

b. Plot these points and connect them with a smooth curve.

c. What is the equation of the curve?

d. Where does it meet the graph of $x + y = 12$?

10. Add to your graph several curves with equations of the form $x \cdot y = P$, where P is an integer, as described below. Draw at least three curves

a. that intersect the graph of $x + y = 12$ in the first quadrant;

b. that intersect the graph of $x + y = 12$ in the second and fourth quadrants;

c. that never intersect the graph of $x + y = 12$.

11. Consider the expression $x^2 + 12x +$ ___. What numbers could you put in the blank to get a trinomial that can be factored? Explain your answer, giving examples.

12. **Report** Summarize what you discovered in this lesson. Concentrate on the question: *How are the points of intersection of constant sum and constant product graphs related to factoring trinomials?* Use examples and illustrate your report with graphs. (The examples given in this lesson involved only positive whole numbers for the sums and products. In your report, you may use negative numbers or zero.)

LESSON 5.5

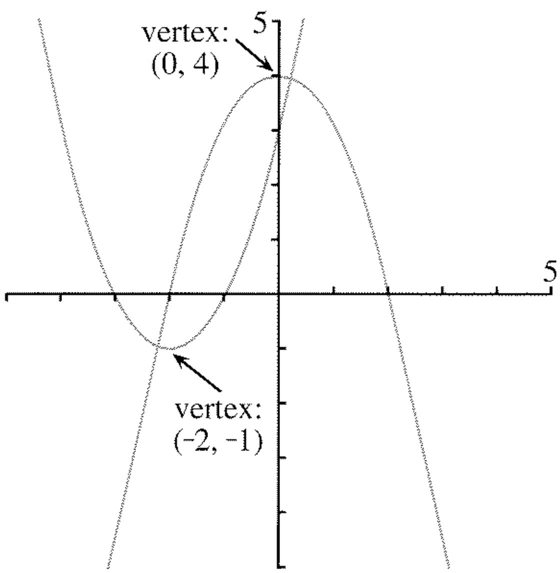
Graphing Parabolas

You will need:
graph paper

Definitions:

- Second-degree polynomial functions are also called *quadratic* functions.
- Graphs of quadratic functions have a special shape called a *parabola.*
- The lowest or highest point on a parabola is called its *vertex.*

Here are two quadratic functions and their graphs. Each one has two x-intercepts and one vertex.



1. What is the y-coordinate of the x-intercepts? What is the x-coordinate of the y-intercept?

2. For each parabola in the figure,
a. what are the x- and y-intercepts?
b. which x-intercept is the vertex closer to?

FINDING INTERCEPTS AND THE VERTEX

3. a. Copy and complete the table of values for the quadratic function $y = x^2 + 2x - 8$. Use at least six values from -5 to 5. Using the format shown will help you avoid making mistakes in computation.

x	$x^2 + 2x - 8$	y
-5	$(-5)^2 + 2(-5) - 8$	7
-4	________	—
...	________	—
4	________	—
5	________	—

b. Use your table to make a graph of the function.
c. Label the intercepts and the vertex.

4. Repeat problem 3 for the function $y = (x + 4)(x - 2)$.

5.
a. Compare your graphs in problems 3 and 4. Explain what you observe.
b. How are the x-intercepts related to the expression $(x + 4)(x - 2)$?

6. The quadratic function $y = x^2 - 6x + 8$ can be written in factored form as $y = (x - 4)(x - 2)$.
a. Make a table of values for this function, including the intercepts and the vertex.
b. Graph the function. Label the intercepts and the vertex.
c. How are the x-intercepts related to the expression $(x - 4)(x - 2)$?
d. How is the y-intercept related to the expression $x^2 - 6x + 8$?

For each problem, 7-10:

a. Write the function in factored form.
b. Make a table of values, including the intercepts and the vertex.
c. Graph the function, labeling the intercepts and the vertex.

7. $y = x^2 - 2x - 3$

8. $y = x^2 + 4x + 3$

9. $y = x^2 - 4x + 3$

10. $y = x^2 + 2x - 3$

11. Write the equation of a quadratic function whose graph would cross the x-axis at (2, 0) and (-3, 0). Explain how you know it will work.

12. Write the equation of a parabola having y-intercept -4. Explain how you know it will work.

13. **Generalization** Consider functions of the form $y = x^2 + bx + c$ that can be factored into $y = (x - p)(x - q)$.

a. How are b, c, p, and q related?
b. How would you find the coordinates of the intercepts?
c. How would you find the coordinates of the vertex?

SMILES AND FROWNS

14. Make a table of values for the quadratic function $y = (x - 4)(x - 1)$ and graph it.

15. Repeat for $y = -(x - 4)(x - 1)$.

16. Compare your graphs from problems 14-15. What is alike about the graphs and what is different? How do their x-intercepts and vertices compare?

17. Write an equation of a quadratic function whose graph satisfies these given conditions.

a. a *smile* parabola having x-intercepts (3, 0) and (-2, 0)
b. a *frown* parabola having x-intercepts (3, 0) and (-2, 0)
c. a *smile* parabola having x-intercepts (-3, 0) and (-2, 0)
d. a *frown* parabola having x-intercepts (-3, 0) and (-2, 0)

18. Explain how you know that your answers to problem 17 are correct. You may check your answers by making a table of values, and graphing.

19. Write the equation of a quadratic function that passes through the origin and (5, 0). Explain.

20. Write an equation of a quadratic function whose graph satisfies the given conditions.

a. a parabola having one x-intercept at (1, 0) and the vertex with x-coordinate 2
b. a parabola having one x-intercept at (1, 0) and the vertex at (2, 1)
c. a parabola having one x-intercept at (1, 0) and the vertex at (2, 2)

HOW MANY x-INTERCEPTS?

21. Graph each of these four quadratic functions on the same axes.

a. $y = x^2 + 6x + 5$
b. $y = x^2 + 6x + 8$
c. $y = x^2 + 6x + 9$
d. $y = x^2 + 6x + 12$

22. Write a paragraph describing and comparing the graphs you drew in problem 21. Which graph or graphs have two x-intercepts? Which have one? Which have none? Could you have predicted this before graphing? Explain.

23. Consider the quadratic function $y = x^2 + 4x +$ ___. Fill in the blank with a number that will give a function whose graph is
    a. a parabola having one x-intercept;
    b. a parabola having two x-intercepts;
    c. a parabola having no x-intercepts.

24. Check your answers to problem 23 by graphing, or explain why you are sure you are correct.

PUZZLES MAKING CHANGE

25. Find the largest number of pennies, nickels, and dimes that you can have and still not be able to make change for a quarter. Explain your answer.

26. Find the largest number of coins you can have and still not be able to make change for a dollar. (Assume that you can have any coins except a silver dollar.) Explain this answer.

PREVIEW ZEROING IN

27. If $ab = 0$, $bc = 0$, and $ac = 1$, what is b?

28. If $abc = 0$ and $bcd = 1$, what conclusion can you draw? Explain.

PUZZLE SQUARE SUMS

29. Arrange the whole numbers from 1 to 18 into nine pairs, so that the sum of the numbers in each pair is a perfect square.

LESSON 5.6

Factors

You will need:

the Lab Gear 

1. Exploration

a. Draw a rectangle whose sides are any two consecutive even numbers, like 4 and 6. Find its area. If the side lengths have to be whole numbers, is it possible to draw a rectangle having the *same area but different sides*? Try this with another pair of consecutive even numbers. Is it possible this time? Do you think it is always, sometimes, or never possible?

b. Does your result change if you use two consecutive odd numbers, like 3 and 5?

c. What about consecutive multiples of 3, like 6 and 9?

SAME AREA, DIFFERENT PERIMETER

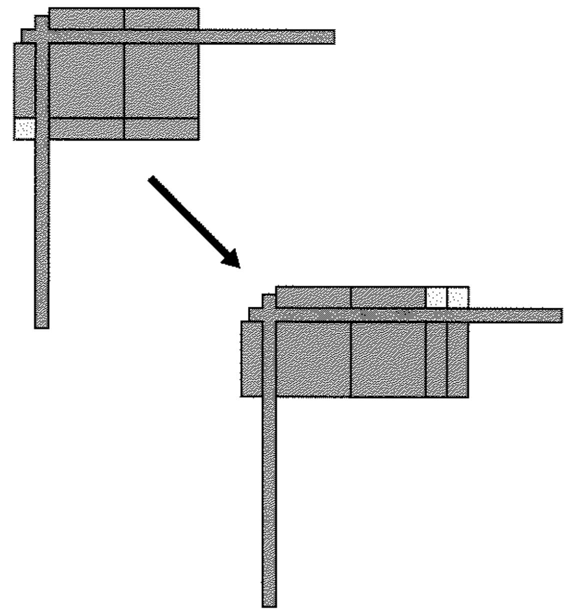
Example: Use the Lab Gear to build a rectangle having a width of $2x$ and a length of $x + 1$.

a. Sketch the rectangle. Label it with an equation of the form *length times width equals area.*

b. Find the perimeter of the rectangle.

c. Rearrange your rectangle into a rectangle having the *same area* but a *different perimeter.*

d. Write another equation of the form *length times width equals area.*



For problems 2-4 below, build a Lab Gear rectangle of the given width and length. Then follow the instructions in parts (a) through (d) in the example.

2. width: $2x$ length: $2x + 2$

3. width: $3x$ length: $3 + x$

4. width: x length: $4 + 4x$

For problems 5-6 follow the instructions in the example, but build at least two rectangles, and three if possible.

5. width: $4 + 2x$ length: $2 + 4x$

6. width: $2 + 2x$ length: $3 + 2x$

RECOGNIZING FACTORS

For each expression, 7-12, write as many different products equal to it as you can. Use only whole numbers. (In some cases, it may be helpful to use the Lab Gear to build rectangles and/or boxes.)

7. 24

8. $6y^2$

9. $(2x + 4)(3x + 6)$

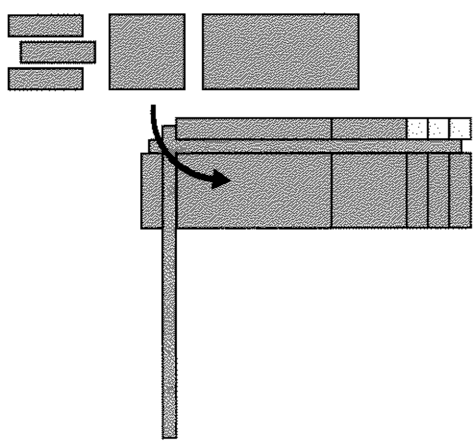
10. $12x^3$

11. $12x^2 + 4x$

12. $2x(6x + 18)$

COMMON FACTORS

Example: As you know, factoring a polynomial can sometimes be modeled by making a Lab Gear rectangle.



$$xy + x^2 + 3x = x(y + x + 3)$$

By multiplying the factors, you get the original polynomial back. Factoring is using the distributive law in reverse.

In this example, we say that x is a *common factor* of all three terms in the original polynomial, because it divides each term evenly. In the case of $2x^3 + 8x^2 + 2x^2y$, the common factors are 2, x, and x^2. In factoring such a polynomial, it is usually best to *take out* the *greatest common factor,* which is $2x^2$.

In the following problems, factor the polynomials by taking out the greatest common factor. Not all are possible.

13. $2x^3 + 8x^2 + 2x^2y$

14. $2x^2 - 6x$

15. $2x^2 + 6x + 1$

16. $3x^2 + 2x + 4xy$

17. $3x^2y - 3xy + 6xy^2$

18. $3y^2 + 9y - 6y^3 + 3x^2y + 6xy^2 + 9xy$

FACTORING COMPLETELY

As you have seen in this lesson, there are often many ways to factor a polynomial. However, there is only one way to factor it *completely.* For example, $(4x + 8)(3x + 9)$ is factored, but to factor it completely you would have to factor 4 out of $(4x + 8)$ and 3 out of $(3x + 9)$.

Factor completely.

19. $(2x + 6)(3x + 6)$

20. $4(x^2 + 5x + 6)$

21. $4x^2 + 40x + 64$

22. $2x^2 + 8x + 8$

23. $3x^2 + 21x + 30$

24. $2x^2 + 26x + 72$

25. $x^3 + 5x^2 + 6x$

LESSON 5.7 Minus and the Distributive Law

You will need:

the Lab Gear 

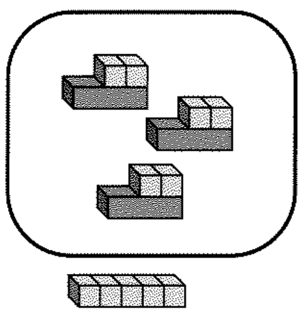
REVIEW ***ORDER OF OPERATIONS***

1. Compare these two expressions, and these two figures.

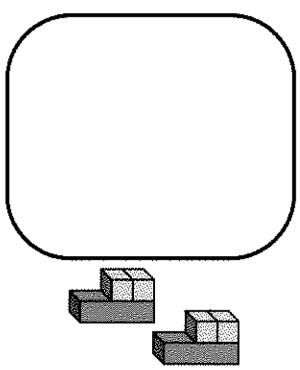
$(5 - 3)(x - 2)$

$5 - 3(x - 2)$

(i)



(ii)



a. Which expression means *multiply $(x - 2)$ by 3 and subtract the result from 5*? (Remember order of operations.)

b. Which figure shows that expression with the Lab Gear?

c. Which expression means *subtract 3 from 5 and multiply the result by $(x - 2)$*?

d. Which figure shows that expression with the Lab Gear?

e. Here are the same expressions, rewritten without parentheses. Which is which?

$11 - 3x$ $\qquad$ $2x - 4$

Write without parentheses.

2. $7 - 3(y - 4)$ $\qquad$ **3.** $(7 - 3)(y - 4)$

4. $(4 - 2)x + 1$ $\qquad$ **5.** $(4 - 2)(x + 1)$

6. $x - 2(x + 1)$ $\qquad$ **7.** $(x - 2)(x + 1)$

8. $(x - 2)x - 1$

If you added another set of parentheses to the expression in problem 8, you would get $(x - 2)(x - 1)$. One way to multiply these binomials is to use the multiplication table format.

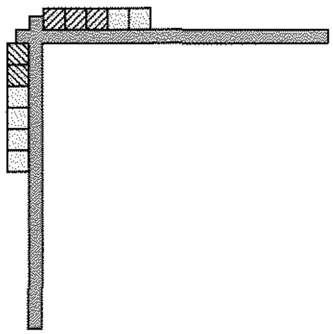
	x	-2
x	___	___
-1	___	___

9. What is the product?

USING THE CORNER PIECE

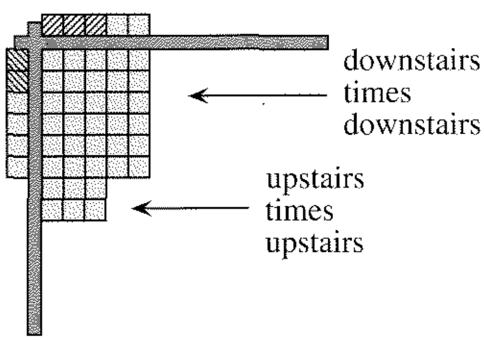
In this lesson, you will learn how to model a product like this with the Lab Gear. You will practice it with numbers before using variables.

Example: In the case of $(6 - 2)(5 - 3)$ set up the problem as shown in the figure. The method you will follow is to *multiply all the blocks on the left side by all the blocks across the top.*

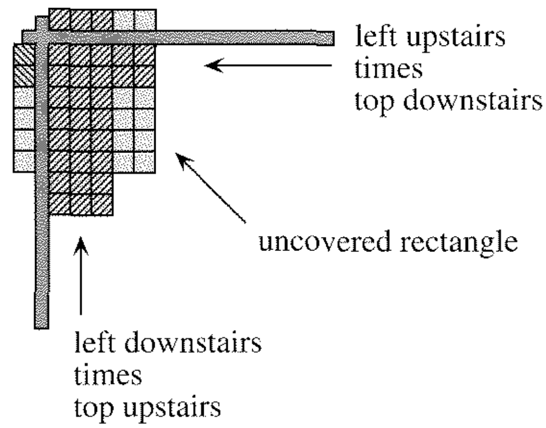


Put the upstairs blocks at the corner of the corner piece.

First, multiply the downstairs blocks. Then multiply the upstairs blocks by each other. Since -2(-3) = 6, a positive number, these blocks must appear *downstairs* somewhere. They will be arranged in a 2-by-3 rectangle. It would be nice to line up the rectangle with its factors, but then it would have to be upstairs, making it -6, which would be wrong. So we can *line it up with only one of the two factors.* Let's choose the -3.



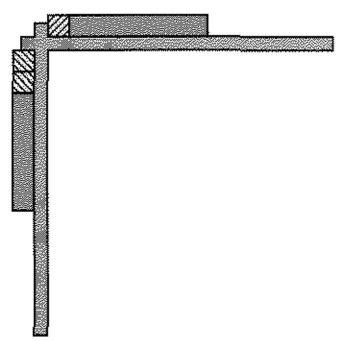
Finally, multiply upstairs blocks on the left with downstairs blocks at the top, and vice versa, placing them as shown.



You can now see that the answer (4 times 2 = 8) is shown by *the uncovered rectangle.*

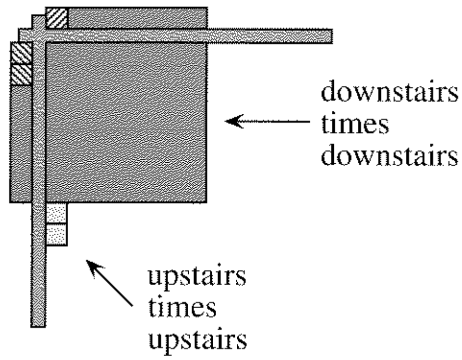
10. Use the corner piece to show the product (5 − 2)(7 − 4).

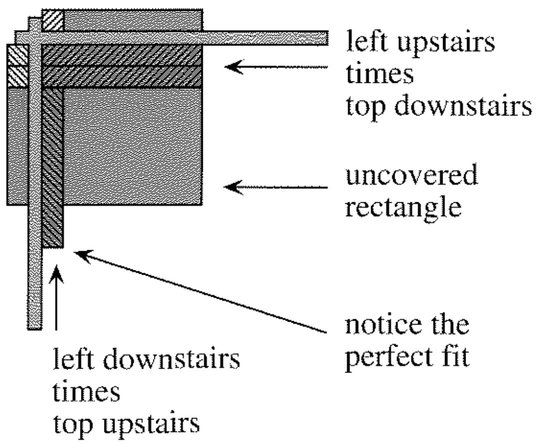
USING VARIABLES



11. Write the polynomials being multiplied.

12. 🔑 Follow the process shown in the following figures with your blocks. Write a brief explanation of each step.





13. Write the dimensions of the uncovered rectangle and the product.

14. Use the Lab Gear to multiply.

a. $(x - 1)(2x - 3)$

b. $(y - 5)(2y - 1)$

15. Use the Lab Gear to multiply.

a. $(y + 1)(y + 5)$

b. $(y - 1)(y + 5)$

c. $(y + 1)(y - 5)$

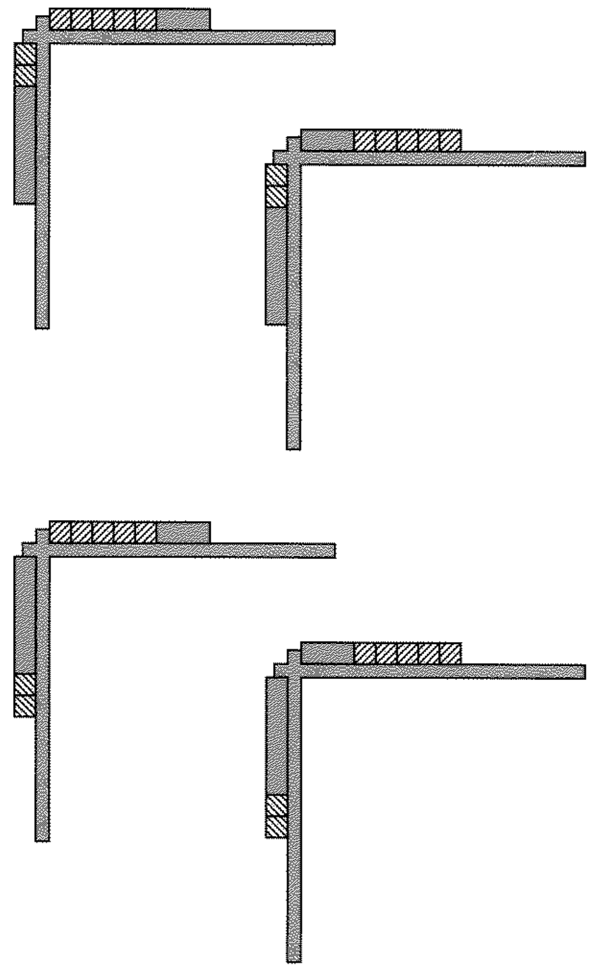
d. $(y - 1)(y - 5)$

16. Use the Lab Gear to multiply.

a. $(2x + 3)^2$ b. $(2x - 3)^2$

c. $(2x + 3)(2x - 3)$

17. This figure shows four ways to set up the multiplication $(y - 2)(y - 5)$. Of those, three will work. Experiment to find out which three. Sketch your solutions.



DISCOVERY ***MAKE A SQUARE***

For each problem, arrange the blocks into a square. Not all are possible. Write an equation relating the side length and area of the square.

18. $x^2 + 6x + 9$

19. $4x^2 + 4x + 1$

20. $x^2 + 8x + 4$

21. $x^2 + 4x + 16$

22. $9x^2 + 12x + 4$

23. $x^2 + 2xy + y^2$

REVIEW ***SOLVING EQUATIONS***

24. Use the cover-up method to solve these equations.
   a. $30 - 3(2x + 1) = 9$
   b. $19 - 2(x + 5) = 1$
   c. $(5 - 3x) - 2 = -3$
   d. $5 - 3(x - 2) = 20$
   e. $(5 - 3)(x - 2) = 10$

REVIEW ***FUNCTION DIAGRAMS FOR CONSTANT PRODUCTS***

For each equation, 25-27:
   a. Make a large function diagram (with number lines ranging from at least -12 to 12), using your calculator to help you find values if needed.
   b. Do all the in-out lines meet in a single point?
   c. Are there any horizontal in-out lines? (In other words, in-out lines where $x = y$.) For what values of x and y?
   d. Follow the y-value with your finger as x changes from -12 to 12. Describe y's motion. (Does it move up or down? Does it ever *jump*? For what values of x does it move fast? Slowly?)

25. $xy = 9$

26. $xy = 8$

27. $xy = -9$

28. x is greater than 1, and $6/x$ is a whole number. What could x be? (Hint: There are more than three solutions.)

LESSON 5.8

Building-Block Numbers

In this lesson, use just whole numbers.

FOOD FOR THOUGHT

1. **Exploration** Eric tried to order 13 chicken nuggets at the fast food store. The employee informed him that he could order only 6, 9, or 20 nuggets. Eric realized he had to decide between ordering $6 + 6 = 12$, or $6 + 9 = 15$. What numbers of nuggets can be ordered by combining 6, 9, and 20? What numbers cannot be ordered? What is the greatest number that cannot be ordered? Explain.

TWO BUILDING BLOCKS

2. You have an unlimited supply of coins. What amounts can be obtained, and what amounts cannot be obtained using only dimes and quarters? Explain.

3. At Albert's Kitchen Supply, cabinets are available in two lengths: 3 feet and 5 feet. By putting cabinets end to end, walls of different lengths can be accommodated. Imagine that kitchens can be any size. What length walls are possible to line exactly with cabinets? What lengths are impossible?

4. In 1958 it cost 4 cents to mail a letter. In 1963 it cost 5 cents. Imagine you have an unlimited supply of 4- and 5-cent stamps. What amounts can you make? What is the largest amount you cannot make?

For each problem, 5-10, using only addition and the building-block numbers given, what numbers can you reach? What numbers can't you reach? If there is one, what is the greatest number you cannot reach?

5. 2, 5 **6.** 7, 11 **7.** 4, 6

8. 5, 12 **9.** 5, 15 **10.** 8, 1

Given the two numbers 7 and 11 and the operation of addition, it is possible to build every number beyond 59. However, with the numbers 4 and 6 there is no limit to the size of numbers that cannot be built.

11. **Generalizations** Suppose you find that for two numbers, *a* and *b*, and the operation of addition, you can build every number beyond a certain number. What can you say about *a* and *b*? Explain, using examples. (Hint: You may need to use the idea of common factors. For example: 4 and 6 have the common factor 2; 5 and 15 have the common factor 5.)

12. Given two numbers, *a* and *b*, such that their greatest common factor is 1, how can you calculate the greatest number that *cannot* be written as a sum of multiples of *a* and *b*? Explain, using examples.

A STRATEGY

In problems 13 and 14 you will investigate the numbers 5 and 6 as building blocks.

13. Write the numbers from 1 to 40 in an array like this.

0	1	2	3	4
5	6	7	8	9
10	11	12	13	14...

a. Circle the multiples of 5. (0 is a multiple of 5.)

b. Circle the numbers that are equal to 6 plus a multiple of 5.

c. Circle the numbers that are equal to 12 plus a multiple of 5.

d. Circle the numbers that are equal to 18 plus a multiple of 5.
e. Circle the numbers that are equal to 24 plus a multiple of 5.

14. What is the largest number that **cannot** be built from 5 and 7? Explain how you know for sure that every number greater than this number can be built.

15. Repeat the same strategy to analyze 5 and 6 as building blocks.

16. Repeat the same strategy to analyze 4 and 7 as building blocks. (This time set up your array with only four columns.)

17. **Generalization** If you were to use the same strategy for numbers a and b, with $a < b$:
a. How many columns should you have in your array?
b. What numbers should you circle first?
c. What numbers should you circle next?
d. What is the smallest number in the last column you circled? (Write this number in terms of a and b.)
e. If you were not able to solve problem 12, try again with the help of this strategy.

DISCOVERY *HOLIDAY MATH*

18. Candles are lit every night for the eight nights of Hanukah. Two candles are lit on the first night, three on the second night, and so on, adding one candle each night. How many candles should be in the boxes of candles sold especially for Hanukah? Explain.

19. Find the words to the song "The Twelve Days of Christmas."
a. Make a sketch or drawing to show what is happening in the song. How many gifts did the singer receive on the twelfth day of Christmas? Explain.
b. The singer received six gifts on the 3rd day. How many gifts did the singer receive on the 4th day? The 5th day? The n^{th} day? Explain.
c. The singer received 22 turtledoves. Find the total number of each other kind of gift that the singer received.
d. Suppose there were n days of Christmas. How many gifts would the singer receive in all? Explain.

PREVIEW *COIN PROBLEMS*

20. You have ten coins. Their total value is \$1.10. How many of each coin do you have? The problem has several solutions. Find as many as you can.

21. Add extra information that makes problem 20 have a unique solution. Explain how you know the solution is unique.

22. Create your own coin problem that has several solutions. Solve your problem.

23. Solve someone else's coin problem.

24. Add extra information to your problem so it will have a unique solution.

DISCOVERY *NEGATIVE STAMPS*

25. You want to mail a letter. It needs 52 cents postage, but all you have are 29-cent stamps: $29 + 29 = 58$. What would be convenient would be to have negative stamps. Then you could put two 29-cent stamps and a minus 6-cent stamp on your envelope, and it would solve your problem. Write a paragraph about this idea. How would the post office "sell" negative stamps? Why do you think they don't do it?

DISCOVERY *PAGE NUMBERS*

26. How many *digits* are used in numbering the pages of this book? Explain how you figured it out.

27. It took 1992 digits to number the pages of a book. Every page was numbered, starting with page 1. How many pages does the book have?

28. Explain how to find out how many digits are needed to number the pages of a book that has n pages, if n is
    a. more than 9, but less than 100;
    b. more than 99, but less than 1000.

THINKING WRITING

5.B Distributing

INSIDE AND OUTSIDE PRODUCTS

Look at this sequence of consecutive integers.

8, 9, 10, 11

- The product of the outside pair is 88.
- The product of the inside pair is 90.
- The difference between the inside product and the outside product is 2.

1. Find the difference between the inside and outside product for each of these sequences.
   a. 4, 5, 6, 7
   b. 10, 11, 12, 13
   c. $10, 10 + 1, 10 + 2, 10 + 3$
   d. $y, y + 1, y + 2, y + 3$
2. What pattern did you notice in problem 1?
3. Look at some sequences of four integers that differ by three. For example, you could try 4, 7, 10, 13. What pattern do you notice in the difference between their inside and outside products?
4. What pattern would you expect to see in the difference of inside and outside products for sequences of numbers that differ by two? What about sequences of numbers that differ by four? Experiment.
5. Find the difference between the inside and outside product for each of these sequences.
   a. $y, y + 2, y + 4, y + 6$
   b. $y, y + 3, y + 6, y + 9$
   c. $y, y + 5, y + 10, y + 15$
   d. $y, y + 5, y + 2 \cdot 5, y + 3 \cdot 5$
   e. $y, y + x, y + 2x, y + 3x$
6. **Report** Write a detailed report describing the patterns you discovered in this lesson. Give examples and show all your calculations. Your report should include, but not be limited to, the answers to the following questions:
   - How is the difference between the inside and outside products related to the difference between numbers in the sequence?
   - How can you use algebra (and/or the Lab Gear) to show that your answer is correct?
   - Does your generalization work for all kinds of numbers? For example, could you choose a sequence made up entirely of negative numbers? What about fractions?

MORE DISTRIBUTIVE LAWS?

You might wonder if there are more distributive laws.

7. Is there a distributive law of exponentiation over addition? If there were, it would mean that $(x + y)^2$ would always be equal to $x^2 + y^2$. It would also mean that $(x + y)^3$ would equal $x^3 + y^3$. Do you think such a law exists? Explain why or why not.
8. Is there a distributive law of multiplication over multiplication? If there were, it would mean that $a(xy)$ would always be equal to $ax \cdot ay$. For example, $2(xy)$ would have to equal $2x \cdot 2y$. Do you think such a law exists? Explain why or why not.
9. **Report** Write a report about distributive laws. Use numerical examples and/or sketches of the Lab Gear. Your report should include a discussion of which of the following laws exist, and why.

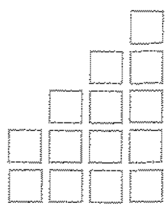
   The distributive law of:
   - multiplication over addition and subtraction
   - division over addition and subtraction
   - exponentiation over addition and subtraction
   - multiplication over multiplication

LESSON 5.9

Staircase Sums

ONE STEP AT A TIME

Here is an example of a kind of arrangement that we'll call a *staircase.* It has 4 steps and the first step is of height 2.



Definition: For this lesson, we will define a *staircase* as a sequence of stacks of tiles in which each stack is one tile higher than the previous stack. *There must be two or more steps in the staircase,* and the first step can be of any height.

1. How many tiles would you need to build each of these staircases?

a. First step: 7 Number of steps: 8

b. First step: 8 Number of steps: 7

c. First step: 6 Number of steps: 9

2. There are two different nine-tile staircases: $2 + 3 + 4$ and $4 + 5$.

a. Find three different 15-tile staircases.

b. Find four different 105-tile staircases.

3. **Exploration** Find every possible staircase with each number of tiles from 2 to 34. Hints:

- Work with other students.
- Keep organized records of your work.
- It is not necessary to draw the staircases.
- Look for strategies: What numbers can be made into two-step staircases? Three-step?
- Look for patterns: What numbers are easiest? What numbers are impossible?

4. The number 10 can be written as the sum of four consecutive numbers.

a. What are these four numbers?

b. If negative integers and zero are allowed, can the number 10 be written as the sum of consecutive numbers in any other way? If so, show how.

5. Show how the number 4 can be written as a sum of consecutive integers if negative numbers and zero can be used.

6. **Generalization** What is the maximum number of consecutive integers that can be used to write the number 17 as a sum? What is the maximum number of consecutive integers that can be used to write the number N as a sum? (Assume N is an integer.) Explain your answer, giving examples.

SUMS FROM RECTANGLES

7. a. On graph paper, sketch the staircase illustrated at the beginning of the lesson. Then make a rectangle by sketching a copy of the staircase *upside down* on the first staircase. (You can also do this by building the staircases with tiles.)

b. What are the length, width, and area of the rectangle?

8. Imagine a staircase having 100 steps, and a first step of height 17.

a. It would be half of what rectangle? (Give the length and width.)

b. How many tiles would you need to build the staircase? Explain how you know.

9. Show how you could find the sum of:

a. the integers from 5 to 55, inclusive;

b. the integers from 0 to 100, inclusive.

GAUSS'S METHOD

Math teachers like to tell a story about Carl Friedrich Gauss. One day in elementary school he was punished by his teacher who asked him to add up all the whole numbers from 0 to 100. Carl immediately gave the answer, to his teacher's amazement. He grew up to be one of the greatest mathematicians of all time.

Gauss's method was to imagine all the numbers from 0 to 100 written from left to right, and directly beneath that, all the numbers written from right to left. It would look like this:

0	1	2	3	4	5	6	7	8	9...
100	99	98	97	96	95	94	93	92	91...

He mentally added each column, getting 100 each time. He multiplied 100 by the number of columns, and did one more thing to get the correct answer.

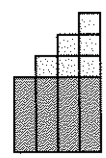
10. Finish Gauss's calculation. Be sure to use the correct number of columns, and to carry out the final step. Did you get the same answer as in problem 9b?

11. What would happen if the numbers to be added started at 1 instead of 0? Obviously, the sum should be the same. Would Gauss's method still give the same answer? Explain.

12. Summary You now know two methods for calculating staircase sums: one involves making a rectangle; the other is Gauss's method. Both methods work well, but it is easy to make mistakes when using them. Write a paragraph explaining how you would use each method to calculate the sum, $5 + 6 + 7 + ... + 89$. Use sketches as part of your explanation. Both methods should give the same answer.

VARIABLE STAIRCASES

You can build staircases with the Lab Gear. This diagram shows $(x) + (x + 1) + (x + 2) + (x + 3)$.



13. In terms of x, what is the sum of $(x) + (x + 1) + (x + 2) + (x + 3)$?

14. Find the sum of $(x) + (x + 1) + (x + 2) + (x + 3)$ if:

a. $x = 4$; b. $x = 99$.

15. Find each sum. Explain how you got your answer.

a. $(x) + (x + 1) + (x + 2) + ... + (x + 26)$

b. $(x + 1) + (x + 5) + ... + (x + 84)$

16. Generalization What is the sum of each staircase?

a. $1 + 2 + 3 + ... + n$

b. $(x) + (x + 1) + ... + (x + n)$

c. $(x + 1) + ... + (x + n)$

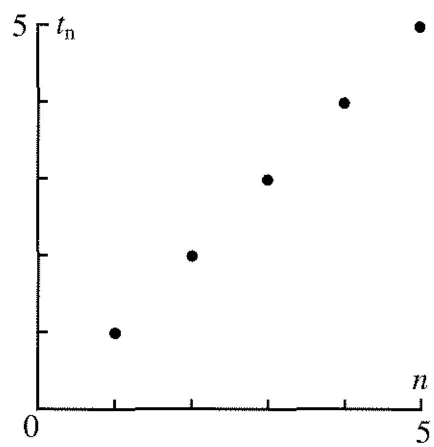
LESSON 5.10 Sequences

Definitions: A *sequence* is an ordered list of numbers, called *terms.* (Notice that this is a new use of the word *term.*) Terms are often indicated with subscripted variables, such as t_1, t_2, or t_n.

GRAPHS OF SEQUENCES

Definition: The *natural numbers* are the numbers we count with: 1, 2, 3, 4, ...

The natural numbers are the easiest sequence of numbers to write using a variable. The first natural number is 1, the second natural number is 2, and so on; $t_1 = 1$, $t_2 = 2$, ... The n^{th} natural number is n, so $t_n = n$. The graph shows the sequence of natural numbers.



1. Graph the first few terms of the sequence below. Does it make sense to connect the dots? Explain.

n	1	2	3	4	...	n
t_n	2	5	8	11	...	$3n - 1$

2. Make a table, and graph the first few terms of the sequence whose n^{th} term is $t_n = 3n + 1$. Compare your graph with the one you drew in problem 1. How are they the same? How are they different?

3. You may remember this sequence.

n	1	2	3	4	...	n
t_n	1	3	6	10	...	?

a. What is the 6th term?

b. What is the n^{th} term?

c. Graph the first few terms. Is your graph a straight line?

GETTING EVEN

4. If 2 is the first even number, 4 the second, and so on, what is the millionth even number? In terms of n, what is the n^{th} even number?

5. Graph the first few terms of the sequence of even numbers. Is your graph a straight line?

n	1	2	3	...	?	n
t_n	2	6	?	...	42	?

6. The n^{th} term in the above sequence is the *sum of the first n even numbers.*

a. What is t_5?

b. Which term has a value of 42?

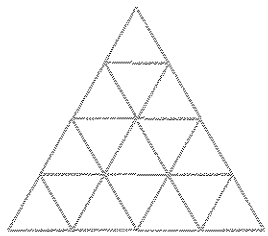
c. Graph the first few terms. Is your graph a straight line?

d. In terms of n, what is the n^{th} term of this sequence?

THAT'S ODD!

7. If 1 is the first odd number, 3 the second, 5 the third, what is the one-hundredth odd number?

8. a. In terms of n, what is the n^{th} odd number?
   b. Graph the first few terms in the sequence of odd numbers.



9. a. Look at the figure. How many unit triangles are in the first row? The second? The third? (Count triangles whether they point up or down.)
   b. If the triangle were extended indefinitely, how many unit triangles would there be in the n^{th} row?

10. a. How many unit triangles are there altogether in the first two rows? The first three rows?
    b. How many unit triangles are in the first n rows?

11. What is the sum of the first two odd numbers? The first three?

12. a. What is the sum of the first n odd numbers?
    b. Graph the first few terms in the sequence of sums of odd numbers.

ARITHMETIC SEQUENCES

Definition: In an *arithmetic sequence,* the difference between consecutive terms is always the same. It is called the *common difference.*

Examples: These are arithmetic sequences.
2, 7, 12, 17, 22 (The common difference is 5.)
5, 8, 11, 14, 17, 20, 23, 26, 29, 32 (The common difference is 3.)

These are not arithmetic sequences.
3, 9, 27, 81
1, -1, 1, -1, 1, -1
4, 9, 16, 25, 49

13. Which of these are arithmetic sequences? For those that are, what is the common difference?
    a. 2, 6, 8, 12, 16, 20
    b. 3, 6, 3, 7, 3, 8
    c. 19, 13, 7, 1, -5, ...
    d. the sequence of even numbers
    e. the sequence of odd numbers
    f. $2, 2 + 9, 2 + 2 \cdot 9, 2 + 3 \cdot 9, 2 + 4 \cdot 9$

14. Make up an arithmetic sequence for another student.

15. Answer these questions about a classmate's sequence.
    a. Is it really an arithmetic sequence?
    b. What is the common difference?
    c. In terms of n, what is the n^{th} term?

16. For each arithmetic sequence, find the common difference, and write the n^{th} term in terms of n.
    a. 2, 7, 12, 22, ...
    b. $2 + 1 \cdot 5, 2 + 2 \cdot 5, 2 + 3 \cdot 5, ...$
    c. $2, 2 + 1 \cdot 5, 2 + 2 \cdot 5, 2 + 3 \cdot 5, ...$

17. Answer the same questions as in problem 15 for:
    a. $y, y + 1 \cdot 5, y + 2 \cdot 5, y + 3 \cdot 5, ...$
    b. $2 + 1 \cdot x, 2 + 2 \cdot x, 2 + 3 \cdot x, ...$
    c. $y + 1 \cdot x, y + 2 \cdot x, y + 3 \cdot x, ...$
    d. $y, y + 1 \cdot x, y + 2 \cdot x, y + 3 \cdot x, ...$

18. **Summary** Explain how to calculate the n^{th} term of an arithmetic sequence, if you know the first term and the common difference. Test your method on several arithmetic sequences.

PREVIEW EQUATIONS

19. For each equation, find values of x_1, x_2, and x_3, that make it true.

a. $(x_1 + x_2 + x_3)/3 = 100$

b. $(x_1 + x_2 + x_3)/3 = 50$

c. $(x_1 + x_2 + x_3)/3 = 20$

d. $(x_1 + x_2 + x_3)/3 = 10$

20. For each equation in problem 19, find another set of values for x_1, x_2, and x_3 that will work.

21. If possible, find a value of x_3 to satisfy each equation.

a. $(15 + 20 + x_3)/3 = 100$

b. $(15 + 20 + x_3)/3 = 50$

c. $(15 + 20 + x_3)/3 = 20$

d. $(15 + 20 + x_3)/3 = 10$

PREVIEW ANOTHER ODD TRIANGLE

1
3 5
7 9 11
13 15 17 19
21 23 25 27 29

22. Look at the array of numbers above.

a. Write the next two rows.

b. Describe how the array is made.

23. a. Look at the middle number in rows that have a middle number. What is the pattern?

b. In rows that do not have a middle number, think of the number between the middle two numbers. What is the pattern?

c. Find the sum of the numbers in each row. What is the pattern?

24.

a. What is the first number in the n^{th} row?

b. What is the last number in the n^{th} row?

c. What is the sum of all the numbers in the first n rows?

LESSON 5.11

Averages and Sums

MEANS AND MEDIANS

You probably know how to find the average of a set of numbers. For example, the ages of the people in Tina's family are 10, 48, 20, 22, and 57. You would find the average age by adding the numbers and dividing by 5.

$$\frac{10 + 48 + 20 + 22 + 57}{5} = 31.4$$

We call this average the *mean*. Another kind of average is the *median*.

Definition: The *median* is the middle of a set of numbers that are in order from least to greatest. To find the median of an even number of numbers, find the two middle numbers and find their mean.

Examples: To find the median age in Tina's family, first write the numbers in ascending or descending order.

10 20 **22** 48 57

The median is **22.**

These are the ages of people in Lana's family: 52, 20, 15, and 53. To find the median, first write the numbers in ascending or descending order.

53 **52** **20** 15

Compute the mean of the middle two numbers: $\frac{52 + 20}{2} = \mathbf{36}$, so the median is 36.

1. Find the mean of the ages in Lana's family. Compare it with the median.

2. Make up a sequence of seven numbers in which

a. the mean is less than the median;

b. the median is less than the mean;

c. the mean and the median are equal.

3. Repeat problem 2 for a sequence of eight numbers.

4. **Exploration** Find some sequences of numbers in which the mean and the median are equal. Work with your classmates and compare your answers. What can you conclude about these sequences? Write a summary of your conclusions, including examples. (At least one example should be an arithmetic sequence, and at least one should not be.)

5. For each example below, make up two sequences that fit the given description.

a. The greatest term is 19, and both mean and median equal 10.

b. There are six terms. The greatest is 25, the mean is 10, and the median is less than 10.

c. There are seven terms. The least is -60, the median is 18, and the mean is less than 18.

d. The mean and the median are both -4. There are nine terms.

6. If possible, make up an arithmetic sequence that fits each description in problem 5. If it's not possible, explain why not.

MEANS AND SUMS

7. Find the mean and the sum of each arithmetic sequence.

a. -2, -14, -26, -38, -50, -62, -74

b. -5, -1.8, 1.4, 4.6, 7.8, 11, 14.2, 17.4

c. 31, 29, 27, 25, 23, 21

d. 17, 20, 23, 26, 29, 32

8. Study your answers to problem 7.

a. In which cases was the mean one of the terms in the sequence?

b. When the mean was not one of the terms in the sequence, how was it related to those terms?

c. How are the number of terms, the mean, and the sum related?

Suppose we wanted to find the sum and the mean of this arithmetic sequence:

3, 9, 15, 21, 27, 33, 39, 45, 51.

Using Gauss's method, write the sequence twice, once from left to right, and then from right to left.

3	9	15	21	27	33	39	45	51
51	45	39	33	27	21	15	9	3

9.

a. Add each column above.

b. Find the mean and the sum of the sequence.

c. How are your answers to (b) related to the sum of each column?

10. Using your results from problem 9, find a shortcut for calculating the sum and the mean of an arithmetic sequence. Try it on the examples in problem 7, comparing your results with your previous answers.

11. Find the sum and the mean of each arithmetic sequence described.

a. The sequence has 15 terms. The first term is 12, and the last term is 110.

b. The first term is -11, and the last term is -33. Each term is obtained by adding -2 to the previous term.

c. The first term is -14, and the difference between consecutive terms is 5. There are 41 terms in the sequence.

d. The first term is 7, and each term is obtained by adding -1.4 to the previous term. There are eight terms in the sequence.

12. Generalization Find the sum and the mean of each arithmetic sequence.

a. The first term is b, and the final term is 5. There are six terms in the sequence.

b. The first term is b, and the final term is f. There are 10 terms in the sequence.

c. The first term is b, and the final term is f. There are n terms in the sequence.

d. The first term is b, and each successive term is obtained by adding d. There are n terms in the sequence.

THEATER SEATS

Seats in a theater are arranged so that there are 35 seats in the front (first) row, 38 in the next row, 41 in the row behind that, and so on, adding three seats each time.

13. How many seats are in the

a. 10^{th} row? b. the n^{th} row?

14. How many total seats are needed if the theater has

a. 26 rows? b. n rows?

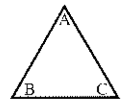
15. How would your answers to questions 13-14 be different if there were 34 seats in the first row?

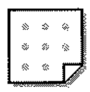
16. Suppose there were 35 seats in the first row, 37 in the next, and so on, adding two seats each time. How would your answers to questions 13-14 be different?

LESSON 5.12

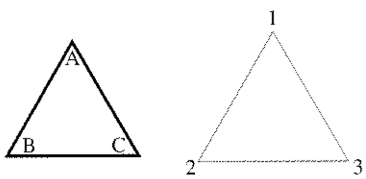
Smooth Moves

You will need:

a cardboard equilateral triangle 

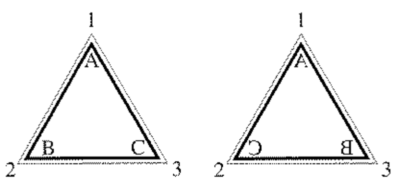
dot paper 

FLIPS AND TURNS

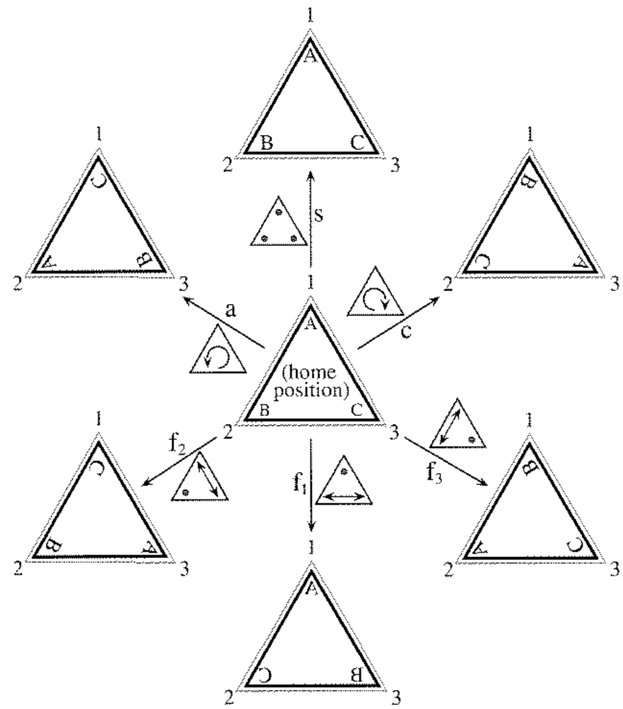


1. a. Write the letters A, B, and C on your triangle, near the vertices. Make sure the same letter appears on both sides of the cardboard at each vertex.

   b. Outline the triangle on a piece of paper, and write the numbers 1, 2, and 3 outside the outline, as in the figure.



There are several different ways you can place the triangle on its outline. The two ways shown in the figure can be written ABC and ACB. ABC is called the *home position.*



∘ means *this corner does not move.*

As you can see on the figure, you can get from the home position to each other position by using one of the following moves.

Turns:

- the clockwise turn (abbreviation: *c*)
- the counterclockwise turn (abbreviation: *a* — short for *anticlockwise*)

To do the turns (also called *rotations*), you do not lift the triangle off the page. You turn it until the triangle fits into the outline again.

Flips:

There are three flips. To do a flip you keep one corner in place and have the other two switch positions. For example, for flip 2 (f_2), you keep corner 2 fixed, and corners 1 and 3 switch positions. (Flips are also called *reflections.*)

Stay:

the *move* that does not move (abbreviation: s)

2. Which corner stays fixed and which changes position
  a. for flip 3 (f_3)?
  b. for flip 1 (f_1)?

Practice the turns and flips, making sure you know what each one does. In this lesson, you will have to execute turns and flips in succession, without going back to the home position in between.

> **Example:** Do f_1, then a. (Such a sequence is simply written f_1a.) If you start at the home position, and do these moves in order, you will end up in the position BAC. (Try it.) But since you could have ended up there *in one move* (f_3), you can write: $f_1a = f_3$.

3. Find out whether $a f_1 = f_3$

4. Simplify. That is, give the one move that has the same result as the given sequence of moves.
  a. $a\,a$ b. $f_1 f_3$ c. $f_3 f_1$
  d. $s f_2$ e. $a\,c$ f. $c\,a$

5. Simplify.
  a. $f_1 f_2 f_3$ b. $a f_1\, a f_2\, a f_3$
  c. $f_1\, a f_2\, a f_3\, a$ d. $c f_1\, c f_2\, c f_3$

6. Figure out a way to write each of the six moves in terms of only f_1 and c.

7. Fill in the blanks.
  a. a ___ $= f_1$ b. ___ $a = f_1$
  c. f_1 ___ $= f_2$ d. ___ $f_1 = c$

THE ALGEBRA OF MOVES

Executing moves in order is an *operation* on triangle moves, just as multiplication is an operation on numbers. The set of six moves, together with this operation, is called the *symmetry group* for the triangle.

8. Make a *multiplication table* for triangle moves. That is, figure out the one move that has the same result as doing the two given moves. Describe any interesting patterns you find in the finished table.

Then...

First...	s	a	c	f_1	f_2	f_3
s	—	—	—	—	—	—
a	—	—	—	f_3	—	—
c	—	—	—	—	—	—
f_1	—	—	—	—	—	—
f_2	—	—	—	—	—	—
f_3	—	—	—	—	—	—

9. For each of the six moves, what move *undoes* it?

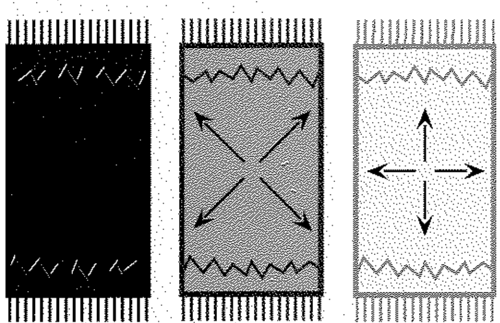
Executing one move (or sequence) repeatedly can be written with *power notation.* For example, f_2^7 means *execute f_2 seven times.*

10. Simplify.
  a. a^{999} b. c^{1000}
  c. f_2^{1000} d. $(af_2)^{1001}$

11. **Project** What flips and turns are possible for another figure, like a rectangle or a square? Write a report on the symmetry group for that figure.

DISCOVERY *MAGIC CARPETS*

Imagine that you can travel from dot to dot on dot paper, using magic carpets such as the ones illustrated in this figure. Carpets cost only $1, plus $1000 per arrow.



De Luxe	Model X	Carpet Plus
$8001	$4001	$4001

Magic carpets move in carpet steps. Each step takes the carpet and its riders to the next dot in the direction of one of the carpet's arrows. Each step takes one second. Carpets do not turn, so that the Carpet Plus cannot move diagonally, and the Model X cannot move horizontally or vertically.

Say you want to go from the origin to (6, 4). Here is a way to get there on each of the three carpets shown.

De Luxe: → → ↗↗↗↗

Model X: ↗↗↗↗↗↘

Carpet Plus: → → → → → → ↑↑↑↑

12. Find another way to get to (6, 4) on each of the three carpets.

13. Compare the advantages and shortcomings of the three carpets. Keep in mind cost, speed, and ability to reach any dot.

14. Project

a. Experiment with various $3001 carpets. What are the advantages and shortcomings of each design? Again, keep in mind cost, speed, and ability to reach any dot. Give a full explanation of your findings.

b. Repeat part (a) for $5001 carpets.

Using the directions North, East, West, and South instead of the arrows, the three examples given above could be written:

De Luxe — E E (NE) (NE) (NE) (NE)

Model X — (NE) (NE) (NE) (NE) (NE) (SE)

Carpet Plus — E E E E E E N N N N,

or even:

De Luxe — $E^2\,(NE)^4$

Model X — $(NE)^5\,(SE)$

Carpet Plus — $E^6\,N^4$.

Since all three paths lead to the same place, we can write:

$$E^2\,(NE)^4 = (NE)^5\,(SE) = E^6\,N^4.$$

In a sense, the last expression is the simplest.

15. What are the rules that allow you to simplify expressions using the N, E, W, S notation? Explain.

THINKING WRITING

5.C Sequences as Functions

A sequence can be thought of as a function. The input numbers are the natural numbers, and the output numbers are the terms. In this assignment, we will study sequences as functions.

Definition: In a *geometric sequence,* each term is obtained from the previous term by multiplying by a constant amount, the *common ratio.*

Examples: These are geometric sequences.

2, 10, 50, 250, 1250

3, 1, 1/3, 1/9, 1/27

For each of the following:

a. Tell whether the sequence is geometric, arithmetic, or neither.

b. If it is arithmetic, find the common difference. If it is geometric, find the common ratio.

1. 5, 1, -3, -7, -11 **2.** -7, 2, 11, 20, 29

3. 1, 1, 2, 3, 5, 8 **4.** 6, 3, 3/2, 3/4, 3/8

5. 25, 5, 1, 1/5, 1/25

6. 1/2, 3/4, 7/8, 5/16, 31/32, 63/64

7. Find the final term of each sequence.

a. a geometric sequence having five terms, common ratio 2, and first term 6

b. an arithmetic sequence having six terms, common difference 9, and first term -4

8. Find the first term of each sequence.

a. an arithmetic sequence having 10 terms, common difference 7, and last term -3

b. a geometric sequence having eight terms, common ratio 1/2, and last term 1/4

9. Graph these arithmetic sequences by graphing the term number (n) on the horizontal axis and the term (t_n) on the vertical axis.

a. 2, -4, -10, -16, -22

b. 2, 8, 14, 20, 26

c. -5, -11, -17, -23, -29

10. Graph these geometric sequences.

a. 2, 6, 12, 24, 48

b. 3, 3/2, 3/4, 3/8, 3/16

c. 1/8, 1/4, 1/2, 1, 2

11. These *mystery sequences* are neither geometric nor arithmetic. Graph them.

a. 5, 8, 13, 20, 29, 40, 53, 68

b. 7, 13, 23, 37, 55

c. -2, 7, 22, 43, 70

12. By looking at the graphs in problem 11, one might think that the sequences are geometric, but it is clear from looking at the numbers that there is no common ratio. However, the numbers do have a special pattern. Find the pattern and describe it.

13. Report Write a report about what you discovered about graphs of arithmetic sequences, geometric sequences, and the mystery sequences in problem 11. Illustrate your report with examples. Your report should include, but not be limited to, answers to the following questions:

- Which sequences have graphs that are straight lines? Which have graphs that are curved? How are the two kinds of curved graphs different?
- For arithmetic sequences, how does the common difference show up in each graph?
- For geometric sequences, what difference does it make in the graph if the common ratio is greater or less than 1?
- What are the graphs of the mystery sequences called?

Essential Ideas

CONSTANT SUMS AND PRODUCTS

1. If possible, write an equation of the form $x + y = S$ such that the graph of the equation

a. lies in the 2nd, 3rd, and 4th quadrants;

b. lies in the 1st, 2nd, and 3rd quadrants;

c. passes through the origin;

d. intersects the x-axis at (-7, 0);

e. contains the point (12, -3.25).

2. A graph has an equation of the form $x + y = S$. Find two more points on the graph if:

a. the point (-3, -5.8) is on the graph;

b. the graph has x-intercept (1/2, 0);

c. the graph has y-intercept (0, -6.5).

3. If possible, write an equation of the form $x \cdot y = P$ such that the graph of the equation

a. lies in the 2nd and 4th quadrants;

b. contains the point (-9, 1/2);

c. passes through (-2.5, -3.5);

d. intersects the graph of $x + y = 16$ at the point (10, 6);

e. passes through the origin.

4. Write one equation of the form $x + y = S$ and one of the form $x \cdot y = P$ such that

a. neither graph passes through the first quadrant;

b. the two graphs intersect at (8, 4) and (4, 8).

THE DISTRIBUTIVE LAW

5. Write an equivalent expression without parentheses. Combine like terms.

a. $2 \cdot (3 + x)$

b. $2 \cdot (3x)$

c. $(6x + 3)(2x - 4)$

d. $(6x \cdot 3)(2x - 4)$

e. $(6x \cdot 3)(2x \cdot 4)$

6. In which part of problem 5 did you use the distributive law to remove parentheses? Explain.

7. Write equivalent expressions without the parentheses. Combine like terms.

a. $-2(9 + x) - x(2 - x)$

b. $-2(9) + x - x(2 - x)$

c. $-2(9 + x) - 2x - x$

d. $-2(9) + x(-2x) - x$

8. In which parts of problem 7 did you use the distributive law to remove parentheses? Explain.

9. Write without parentheses. Combine like terms.

a. $(x + 3)(x + 5)$

b. $(x + 3)(x - 5)$

c. $(x - 3)(x - 5)$

d. $(x - 3)(x + 5)$

10. Divide.

a. $\frac{6y^2 + 4xy}{2y}$

b. $\frac{4x + 4}{4}$

FACTORING

11. Multiply $(2x - 7)(3x + 5)$.

12. Factor $6x^2 - 11x - 35$.

13. a. Fill in the blank with a whole number so that the trinomial $x^2 + 9x +$ ___ can be factored as a product of binomials. Write the factored form.

b. How many different integer answers are there for part (a)? Find all of them. (Don't forget negative integers.)

14. a. Fill in the blank with an integer so that the trinomial $x^2 +$ ___ $x + 18$ can be factored as a product of binomials. Write the factored form.

b. How many different integer answers are there for part (a)? Find all of them. (Don't forget negative integers.)

15. Factor completely.

a. $(2x + 8)(x^2 + 2x)$

b. $2yx^2 + 12yx + 16y$

c. $x^3 + 6x^2 + 8x$

16. How many x-intercepts does each parabola have? Explain.

a. $y = x^2 + 12x + 20$

b. $y = x^2 + 12x + 36$

c. $y = x^2 + 12x + 49$

d. $y = x^2 - 12x + 36$

17. In problem 16, find the coordinates of:

a. the y-intercept;

b. the x-intercept(s), if any;

c. the vertex.

SEQUENCES

18. If you were to plot these sequences (with n on one axis and t_n on the other axis), for which one(s) would the points lie in a straight line? Explain how you know.

a. 3, 3.5, 4.5, 5.5, 6.5

b. -1, -10, -19, -28, -37, -46

c. 1/2, 1/4, 1/8, 1/16, 1/32

d. 4, 7, 11, 16, 22, 29

PYRAMIDS

A pyramid is made by stacking rows of blue, red, and yellow blocks. There are 100 blocks in the bottom (first) row, 98 in the next row, and so on, with 2 fewer blocks in each successive row. The bottom row is blue, the next row is red, the third row is yellow, and so on, continuing the pattern.

19. Make a sketch or schematic drawing of what you think the pyramid might look like. Write about any patterns you notice.

20. How many rows of blocks are there?

21. How many rows of each color are there?

22. How many blocks are in the 10^{th} row? 11^{th} row? n^{th} row? Top row?

23. What color is the 10^{th} row? What color is the top row?

24. There are 30 blocks in a row. Which row is it?

25. Given the number of a row (5^{th}, 10^{th}, 20^{th}, etc.) can you give its color? Explain the pattern.

26. Given the number of blocks in a row, can you give its color? Explain the pattern.

27. How many blocks in all are needed to build the pyramid?

28. How would your answers to questions 19-27 be different if there were 50 blocks in the bottom row?

29. Suppose four colors were used instead of three. Would any of your answers to problems 19-27 be different? Explain.

30. **Report** Summarize and explain the patterns you noticed in the above problems. What generalizations can you make?

PRACTICE

REVIEW

THE DISTRIBUTIVE LAW

Simplify each pair of expressions.

1. a. $4 + 2[s - 6s - 1)]$
b. $(4 + 2)[s - 6(s - 1)]$

2. a. $4x - 2x[x - 6(x + 1)]$
b. $4x - x[x - 6(x + 2)]$

EQUATIONS

3. The solution to this equation is 6:

$$5x - 1 = 29.$$

a. Change one number in the equation so that the solution will be 5.

b. Change the coefficient of x so that the solution to the equation will be 15.

PARENTHESES

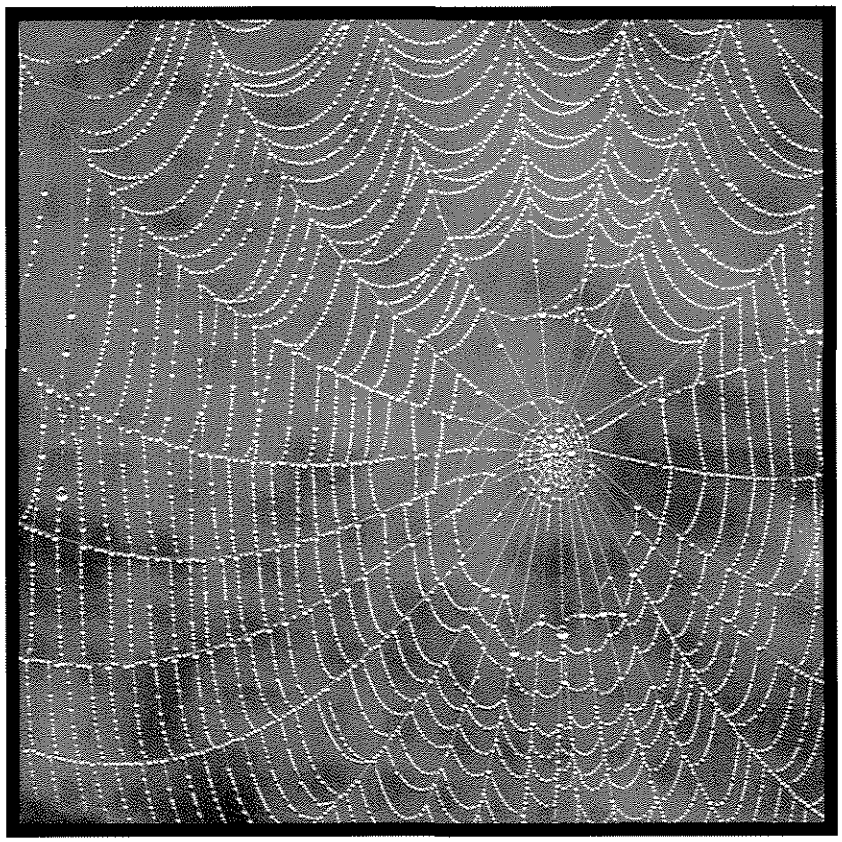
4. Simplify. Compare your answers for (a) and (b).
a. $10 - 5 - 3 + 2$
b. $10 - (5 - 3 + 2)$

5. What is the smallest number you can get by inserting parentheses in the first expression? The largest number? Explain, showing your work.

6. a. Make up an expression containing three terms whose value depends upon where the parentheses are placed. Find all the possible values.

b. Make up an expression containing three terms whose value does not depend upon where the parentheses are placed.

CHAPTER 6



The outward spiral path of a spider web

Coming in this chapter:

Exploration A movie discount card, valid for three months, costs \$$T$. With the card, it costs only \$$D$ to attend a movie, instead of \$5. How many movies would you have to see in three months in order to save money with the discount card? Does your answer depend on T, D, or on both?