Working Backwards

LESSON 3.1

Instant Riches

AMAZING OPPORTUNITY!

The following ad appeared in the school paper.

> Amazing investment opportunity at Algebank! Double your money instantly! Invest any amount! No amount is too small. Our bank will *double* the amount of money in your account every month. Watch your money grow!
>
> A service charge of $100 will be deducted from your account at the end of every month.

1. **Exploration** Do you think this is a good deal? Why or why not? Use some calculations to back up your opinion.

2. Reg was interested in this investment. After calling to make sure that the $100 fee would be deducted *after* his money was doubled, he decided to join. However, after his service charge was deducted at the end of the fourth month, he discovered that his bank balance was exactly $0! How much money did he start out with? Explain your answer.

Three other students invested their money. Gabe started with $45, Earl with $60, and Lara with $200. The figure shows a way to keep track of what happened to Lara's investment.

Month:

$$\overset{0}{200} \xrightarrow{\cdot 2} 400 \xrightarrow{-100} \overset{1}{300} \xrightarrow{\cdot 2} 600\ldots$$

3. a. Use arrows in this way to show what happened to Lara's, Gabe's, and Earl's investments for the first five months.
   b. Give advice to each of these students.

RUNNING OUT OF MONEY

4. Bea joined the plan, but discovered after one month that she had an account balance of exactly $0. How much money had she invested?

5. Lea discovered that she had an account balance of exactly $0 after two months. What was her initial investment?

6. Rea had an account balance of exactly $0 after three months. How much money did she start out with?

7. Summarize your answers to problems 4-6 by making a table like the one below. Then extend the table to show up to at least ten months.

Months to Reach a Zero-Dollar Balance

Months	Amount Invested
1	
2	
...	

8. Describe the pattern in your table.

GAINING AND LOSING

9. Mr. Lear joined the plan, but discovered that at the end of every month he had exactly the same amount of money as when he started. How much money is it? Explain how that happened.

10. Algebank sends its customers statements quarterly (every three months). Several students were comparing their statements at the end of the first quarter. One had \$50, another had \$100, and a third had \$150 in the account.

a. What will happen to each student? Will all of them eventually gain money? What will their next quarterly statements look like? Explain.

b. Explain how you can figure out how much money each of them started with.

11. Find two initial investment amounts that differ by \$1, such that one of them will make money in this plan, and the other will lose money. How far apart will the amounts be in six months? Explain.

12. Report You have been asked to write an article on Algebank's investment plan for the Consumers' Guide column in the school paper. Write an article giving general advice to people wanting to join this plan. Describe the plan clearly and explain the pros and cons of joining it. Who will benefit from the plan? Who will lose in the long run? Explain, giving some examples. Make your article interesting, eye-catching, and readable.

13. Generalization Use what you have learned in this lesson to answer the following questions about plans with similar policies, but different numbers.

a. Give advice to people wanting to join a plan, if their money is *tripled* every month and the service charge is \$100.

b. Give advice to people wanting to join a plan if their money is doubled every month but the service charge is \$200.

14. Suppose Algebank were to deduct the service charge *before* doubling the money. How would this change your answers to problems 12 and 13b?

15. Describe another possible investment scheme and give advice to people about who should join and who should not.

LESSON 3.2

Two Negatives

You will need:

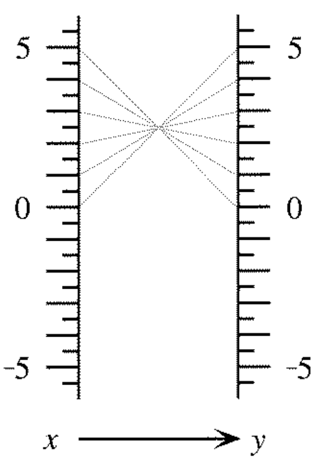
- graph paper
- function diagram paper

1. **Exploration** Many people have heard the rule that *two negatives make a positive.* Investigate to decide whether this rule is always, sometimes, or never true when you *add* two negative numbers. Explain, giving examples. Then repeat your investigation for *subtracting, multiplying,* and *dividing* two negative numbers. Write a brief summary explaining your conclusions.

2. What does *not unilliterate* mean? What about *not uninteresting*? Look up *irregardless* in a dictionary.

SUBTRACTION

3. This function diagram represents a function of the type $y = b - x$. What is the value of b?



4. Make an in-out table for the in-out lines shown on the function diagram.

5. Copy the function diagram. Extend the table and the function diagram for negative values of x.

6. If you know the values of b and x, how can you calculate $b - x$ *by using addition?* Explain, using examples.

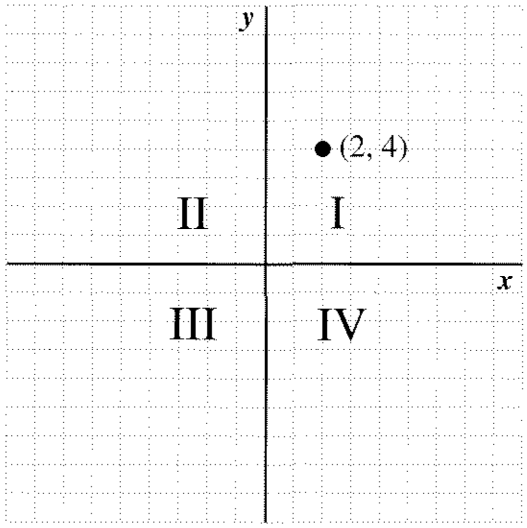
THE CARTESIAN COORDINATE SYSTEM

When you draw horizontal and vertical axes and plot points you are using a *Cartesian coordinate system.* It is named after the French mathematician and philosopher René Descartes. He is credited with bringing together algebra and geometry by using graphs to make geometric representations of algebraic equations.

An important skill in algebra is predicting what the graph will look like from the equation, or what the equation will be from the graph.

You should know the vocabulary of the Cartesian coordinate system.

- The horizontal number line is the *x-axis.*
- The vertical number line is the *y-axis.*
- The numbers (x, y) associated with a point are the *coordinates* of the point.
- The axes divide the coordinate system into four parts, called *quadrants.*
- The quadrants are numbered counterclockwise, as shown. In the first quadrant, the coordinates of every point are both positive.
- The point where the axes cross is called the *origin.* The coordinates of the origin are (0, 0).



7. In which two quadrants does a graph lie if
   a. the second coordinate is always positive?
   b. the first coordinate is always positive?
   c. the two coordinates always have the same sign (both positive or both negative)?

8. What can you say about the signs of x and/or y if you know that (x, y) is in either
   a. the third or the fourth quadrant?
   b. the second or the fourth quadrant?
   c. the second or the third quadrant?

9. If a point is on the x-axis, what is its y-coordinate? If a point is on the y-axis, what is its x-coordinate?

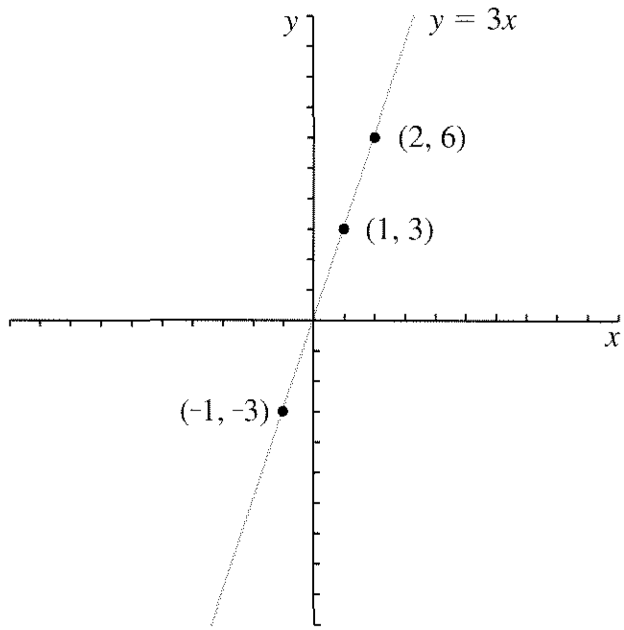
Important: Zero, 0, is neither positive nor negative.

10. Make a Cartesian graph for the function from problem 3, using the in-out table you made in problems 4 and 5.

11. Look at the part of the graph where the y-values are greater than 5. What are the x-values there? Explain what this says about *two negatives*.

MULTIPLICATION

The graph below shows the function $y = 3x$. The y-coordinate is always three times the x-coordinate. Three points are labeled.



12. a. List three more (x, y) pairs that would be on the graph above, including at least one negative and one fractional value for x.
    b. In which two quadrants does the graph lie?
    c. In each (x, y) pair, how are the signs of the x-coordinate and the y-coordinate related?

13. This problem is about the function $y = -3x$.
    a. Make a table of at least six (x, y) values for this function. Use negative numbers and fractions as well as positive whole numbers.
    b. Write the multiplication fact that is represented by each (x, y) pair in your table.
    c. Use your table to make a graph of the function $y = -3x$.

d. In which two quadrants does the graph lie?

e. In each (x, y) pair in your table, how are the signs of the x-coordinate and the y-coordinate related?

14. a. Make a function diagram for the function $y = -3x$.

b. On the diagram, see how the signs of x and $-3x$ are related. When x is negative, what can you say about $-3x$?

15. What is the sign of the answer (positive or negative) when you

a. multiply a negative number and a positive number?

b. multiply two negative numbers?

c. multiply three negative numbers?

16. What is the sign of the answer? (You do not need to find the answer.)

a. $(-5)(-4)(-3)(-2)(-1)(0)(1)(2)(3)(4)(5)$

b. $(-9)(-87)(-7.65)(-43210)$

c. $(-9)^9$ d. $(-99)^{99}$

MULTIPLYING BY -1

Match each function diagram 17-19 with one or more functions from this list.

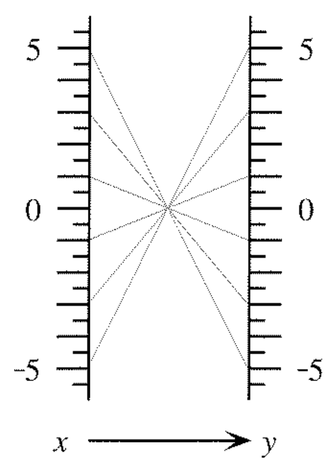
a. $y = 0$ b. $y = x$

c. $y = x + 0$ d. $y = 1 \cdot x$

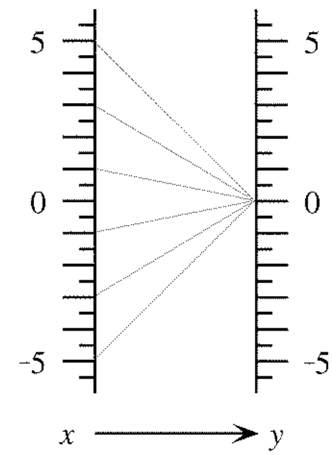
e. $y = -x$ f. $y = -1 \cdot x$

g. $y = 0 \cdot x$ h. $y = 0 \cdot x^2$

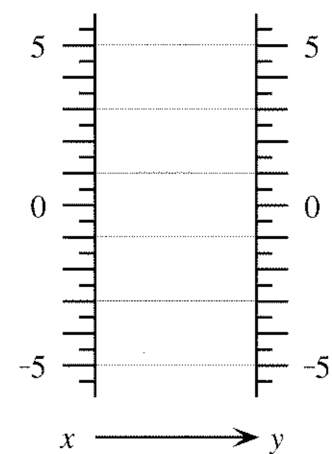
17.



18.



19.



20. *Multiplying x by -1 is the same as taking the opposite of x.* Explain.

21. **Generalization** Explain each step of this calculation.

a. $(-x)(-y) = (-1)(x)(-1)(y)$

b. $= (-1)(-1)(x)(y)$

c. $= (1)(x)(y) = xy$

22. Simplify $(-a)(b)(-c)(-d)$ by the same method.

23. Find each product.

a. $-3 \cdot 5y(-x)$

b. $(-2y)(-3x)(-4)(12xy)$

c. $(-1.3x)(-7x^2)$

d. $(-3x)^2$

e. $(-3x)^3$

LESSON 3.3

More on Minus

You will need:

the Lab Gear 

1. **Exploration** Choose several numbers and investigate the following questions. Write an explanation, using variables, of what you discover. What is the result when you
   a. add a number to its opposite?
   b. subtract a number from its opposite?
   c. multiply a number by its opposite?
   d. divide a number by its opposite?

MINUS AND THE DISTRIBUTIVE LAW

For each problem below:

- Use the Lab Gear to model the first expression on the left side of the workmat.
- If possible, simplify the expression by adding zero and removing matching blocks. Get all blocks downstairs.
- Then decide which of the expressions a, b, c, or d is equal to the given expression. Setting up each one in turn on the right side of the workmat may help. Explain your answers.

2. $x - (5 + 2x)$
   a. $x - 5 + 2x$ b. $x - 5 - 2x$
   c. $x + 5 + 2x$ d. $x + 5 - 2x$

3. $2x - (-4 + 3x)$
   a. $2x - 4 + 3x$ b. $2x - 4 - 3x$
   c. $2x + 4 + 3x$ d. $2x + 4 - 3x$

4. $3y + (5 - 2y)$
   a. $3y - 5 + 2y$ b. $3y - 5 - 2y$
   c. $3y + 5 + 2y$ d. $3y + 5 - 2y$

5. $x - (7 - 2y)$
   a. $x - 7 + 2y$ b. $x - 7 - 2y$
   c. $x + 7 + 2y$ d. $x + 7 - 2y$

6. $6x - (-3 - x)$
   a. $6x - 3 + x$ b. $6x + 3 + x$
   c. $6x - 3 - x$ d. $6x + 3 - x$

7. Write an equivalent expression without parentheses.
   a. $2x^2 - (4 - x - x^2)$
   b. $(2x^2 - 4) - (x - x^2)$
   c. $(y - 5) - 3x - 2$
   d. $y - 5 - (3x - 2)$

8. Write an expression containing at least one pair of parentheses that is equivalent to the given expression. (Do not put parentheses around the whole expression, or around a single term.)
$$3x^3 - 6x + 2 - 5y$$

9. Compare your answers to problem 8 with your classmates. Try to find several different correct answers.

A minus sign preceding parentheses tells you to subtract or take the opposite of everything in the parentheses. Writing an equivalent expression without parentheses is called *distributing the minus sign.*

10. **Summary** Explain how to distribute a minus sign. Use examples.

11. Write an equivalent expression without parentheses.
    a. $-(r + s)$ b. $-(-r + s)$
    c. $-(r - s)$ d. $-(-r - s)$

12. Write an equivalent expression without parentheses.
    a. $-1(r + s)$ b. $-1(-r + s)$
    c. $-1(r - s)$ d. $-1(-r - s)$

You can see from these problems that distributing a minus sign is really just distributing -1.

ADDING THE OPPOSITE

Find the expression that must be added or subtracted. It may help to use the Lab Gear.

13. a. $3x^2 + (-5x) + \_\_\_ = -(5x + x^2)$
b. $3x^2 + (-5x) - (\_\_\_) = -(5x + x^2)$

14. a. $-2xy + x + \_\_\_ = 6xy - 2x$
b. $-2xy + x - (\_\_\_) = 6xy - 2x$

15. a. $-12 + 4yx + \_\_\_ = 7xy - 15$
b. $-12 + 4yx - (\_\_\_) = 7xy - 15$

16. Compare your answers to parts (a) and (b) in problems 13-15. How are they related? Explain.

17. **Generalization** Problems 13-15 illustrated the following fact: *Subtracting is the same as adding the opposite.* For each subtraction, write an equivalent addition.
a. $y - (-x)$
b. $y - x$
c. $-y - x$

18. Find the sign of the answer. (You do not need to find the answer.)
a. $1646 - (-2459)$
b. $-2459 - 1646$
c. $-1646 - (-2459)$
d. $2459 - (-1646)$
e. $-1646 - (2459)$

19. Simplify each expression.
a. $6 - (-5)$
b. $-5 - (-7)$
c. $-21 - (-3x) + 15$
d. $-2x - (-12x) - 5xy$

20. Find each difference.
a. $2y - 7y$
b. $3xy - (-2xy)$
c. $-x^2 - 4x^2$
d. $2xy - 2x$

REVIEW *AREA AND MULTIPLICATION*

21. What is the other side of a rectangle, if one side is x and the area is
a. $5x$?
b. x^2?
c. $x^2 + 2xy$?
d. $x^2 + 2xy + 5x$?

The following equations are of the form *length times width = area of the rectangle.* Fill in the blanks. You may use the Lab Gear to help you. If you do, remember to use *upstairs* for minus and to build a figure with an *uncovered rectangle* of the required dimensions in the corner piece.

22. $x \cdot \_\_\_\_\_ = xy - x^2$

23. $(y - 2) \cdot \_\_\_\_\_ = 5y - 10$

24. $(\_\_\_\_\_ - 3) \cdot x = 2xy - 3x$

25. $2x \cdot \_\_\_\_\_ = 2xy + 4x^2 - 10x$

Use the Lab Gear for these.

26. $(x + \_\_\_\_\_)(y - 5) = xy + 5y - 5x - 25$

27. $(y - 1) \cdot \_\_\_\_\_ = xy + 5y - x - 5$

28. $(y + 2)(y - 1) = \_\_\_\_\_$
(Simplify.)

29. $(y - 1) \cdot \_\_\_\_\_ = y^2 + 4y - 5$
(Hint: Study problem 28.)

DISCOVERY *A SUBSTITUTION CODE*

This message has been coded by a *simple substitution code.*

Rules:

- Each letter is always replaced by the same letter throughout the message.
- No letter is ever replaced by itself.

QEB NRIB CLN QEFP GFKA LC TLAB FP QEHQ BHTE IBQQBN FP HISHUP NBMI-HTBA OU QEB PHJB IBQQBN QENLRDELRQ QEB JBPPHDB.

30. Try to break the code. (Copy the message carefully, leaving blank space between the lines. If you have a guess for a letter, enter it every place that letter appears. For clarity, use lower-case letters for your solution, and capitals for the coded message. Use a pencil and an eraser. Hint: The first word is a very common three-letter word.)

PREVIEW *MAKE A RECTANGLE*

31. For each problem make a Lab Gear rectangle having the given area. Write a multiplication equation.

a. $x^2 + 9x + 8$

b. $x^2 + 6x + 8$

LESSON 3.4

Algebra Magic

You will need:

the Lab Gear 

MAGIC TRICKS

1. **Exploration** A magician asked everyone in the audience to think of a number. "Don't tell your number to anyone," she said. "Now do the following things to your number.

Step 1: Add the number to one more than the number.
Step 2: Add 7 to the result.
Step 3: Divide by 2.
Step 4: Subtract the original number.
Step 5: Divide by 4.

When you are finished, you should all have the same number"

What was the number, and how did the magician know it would be the same for everyone?

2. Try the following algebra magic problem. Record your result and compare it with others in your group. Do you all get the same answer, or does your answer depend on the number you started with?

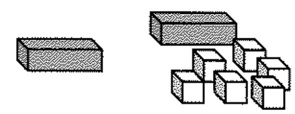
1) Think of a number.
2) Multiply the number by 3.
3) Add 8 more than the original number.
4) Divide by 4.
5) Subtract the original number.

3. Do the same trick, but change the final step to *subtract* 2. Compare answers with your group members again. Are they the same or different? Explain.

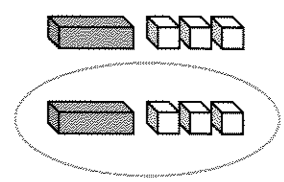
LAB GEAR MAGIC

The following trick has been modeled with the Lab Gear.

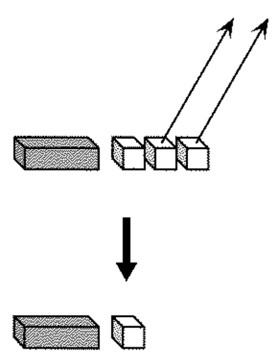
1) Think of a number.



2) Add 6 more than the original number to the number.



3) Divide by 2.
4) Subtract 2.



4. a. In this magic trick, do you think everyone should end up with the same or different answers? Explain.

b. How will a person's answer be related to his or her original number? Explain.

5. Do the following magic trick with the Lab Gear. Start with an x-block, which represents the number a person chose. Sketch each step and write it algebraically.

1) Start with any number.
2) Multiply the number by 4.
3) Add 5.
4) Subtract 1.
5) Divide by 4.
6) Subtract one more than the original number.

Should everyone have the same result? If yes, what is it?

REVISING MAGIC TRICKS

6. Change the magic trick in problem 5 by reversing the order of Steps (3) and (4). Do you get the same answer as you did before? Explain.

7. Change the magic trick in problem 5 by reversing the order of Steps (2) and (3). Was this harder or easier than reversing Steps (3) and (4)? Explain.

8. Change the last step in problem 5 so that everyone ends up with the number they started out with.

9. Do the following algebraic magic trick. Which steps can you reverse without changing the result? Why?

1) Think of a number.
2) Subtract 7.
3) Add 3 more than the number.
4) Add 4.
5) Multiply by 3.
6) Divide by 6.

You should end up with the original number.

INVENTING MAGIC TRICKS

The following trick has one step missing.

1) Think of a number.
2) Take its opposite.
3) Multiply by 2.
4) Subtract 2.
5) Divide by 2.
6) ?????

10. Use the Lab Gear to model the first five steps of this trick. Use y to represent the original number. Then translate each step into an algebraic expression. Compare your result after step (5) with your classmates' answers.

11. Decide what step (6) should be, so that the given condition is satisfied.

a. The final result is one more than the original number.
b. The final result is the opposite of the original number.
c. The final result is always zero.
d. The final result is always -1.

12. For each of these conditions, (a-d), make up an algebra magic trick with at least five steps.

a. The final result is the original number.
b. The final result is 2, regardless of what the original number was.
c. The final result is the same, whether you do the steps backward or forward.
d. The trick uses all four operations (multiplication, division, addition, subtraction).

13. Summary Choose one of the tricks you wrote in problem 12. Test your trick with three numbers, including a negative number and a fraction. Show your work. Use algebra to explain the trick.

THINKING WRITING

3.A Secret Codes

First we will use functions to create codes. Later we will use functions to break codes. Assign a number to each letter of the alphabet. A is 1, B is 2, and so on.

A	1	H	8	O	15	V	22
B	2	I	9	P	16	W	23
C	3	J	10	Q	17	X	24
D	4	K	11	R	18	Y	25
E	5	L	12	S	19	Z	26
F	6	M	13	T	20		
G	7	N	14	U	21		

Definition: The text of a message, before it is encoded, is called the *plaintext.*

The easiest code works by replacing each letter by one that follows it at a certain distance in the alphabet. For example, A (letter 1) is replaced with H (letter 8), B (2) is replaced with I (9), and so on. The function used in this example is $y = 7 + x$, where x is the number of the plaintext letter, and y is the number of the coded letter.

If the number of the coded letter is greater than 26, subtract 26 from it. For example, V's number is 22, $22 + 7 = 29$, $29 - 26 = 3$, so the code letter for V is C.

1. Copy and complete this table to show the $y = 7 + x$ code.

Plaintext	Code
A	H
B	I
C	...

2. Use $y = 7 + x$ to encode the words `smile, juggle, dance, puzzle.`
3. Choose a number, b, and use $y = b + x$ to encode a message for a classmate. (Let the classmate know the value of b so he or she will be able to decode the message quickly.)
4. Decode the following message, which has been encoded with $y = 10 + x$.
`DRSC COXDOXMO ECOC RKVP DRO VODDOBC SX DRO KVZRKLOD.`
5. Find the function that would decode the message in problem 4. Check your answer by actually using it on `DRSC`, and making sure it gives the expected plaintext.
6. a. Use the function $y = 27 - x$ to encode these names.
`Bernard, Carol, Ellen, Peter`
   b. Describe in words the code obtained from this function.
7. a. Encode your name with $y = 30 - x$.
   b. Now take the answer to (a) and encode it with $y = 30 - x$ again.
   c. Comment on the result in (b).
8. a. Encode the word `bilingual` with $y = 8 - x$ and then with $y = x - 8$. Do you get the same answer? Explain.
   b. Find a decoding function for each function in part (a).
9. **Report** In this lesson you learned about two kinds of coding functions. Some look like $y = 7 + x$, and others look like $y = 8 - x$. Write a report on how to decode messages coded by each kind of function and also by functions like $y = x - 8$. Give examples using other numbers for each of the three kinds of functions. Mention any special numbers. (For example, what happens when $y = x + 26$?)

LESSON 3.5

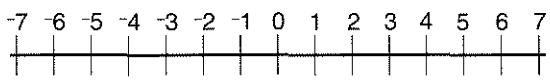
Introduction to Inequalities

You will need:

the Lab Gear 

WHICH IS GREATER?

You can tell which of two numbers is greater by their positions on the number line.



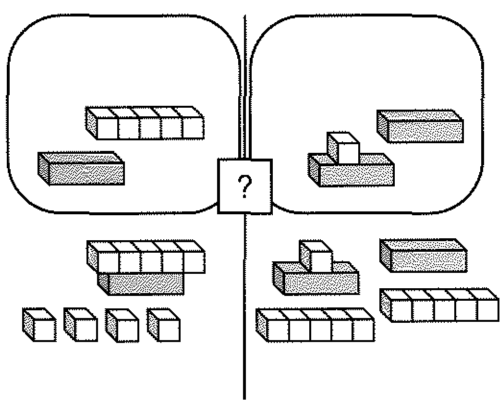
The number that is greater is farther to the right. The number that is less is farther to the left.

Notation: The symbol for *less than* is <. For example, $-5 < 3$, $0 < 7$, and $-6 < -2$. The symbol for *greater than* is >. For example, $6 > 3$, $0 > -2$, and $-5 > -9$.

1. Use the correct symbol.

a. -5 ? -7 b. -5 ? -1

This workmat shows two expressions.
$x + 4 - 5 - (x + 5)$ and $10 + 2x - 1 - (2x - 1)$



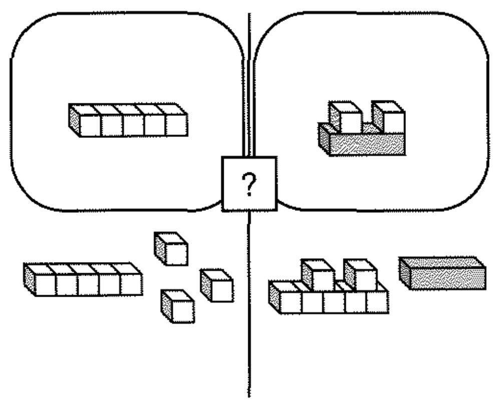
Which is greater? The question mark shows that this is unknown.

2. Put out blocks to match the figure. Simplify both sides. Write an expression for the blocks that remain on the left side. Write an expression for the blocks on the right side. Which side is greater? Show your answer by writing the correct *inequality sign* between the two expressions.

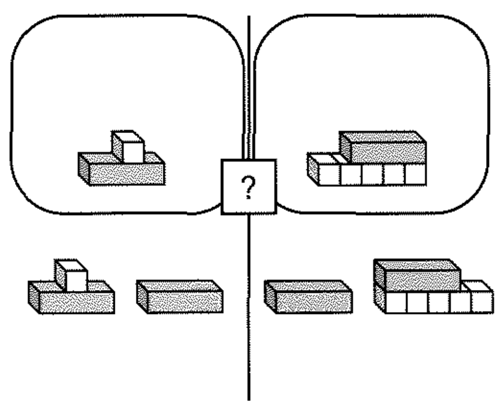
For each problem, put out blocks to match the figure, and

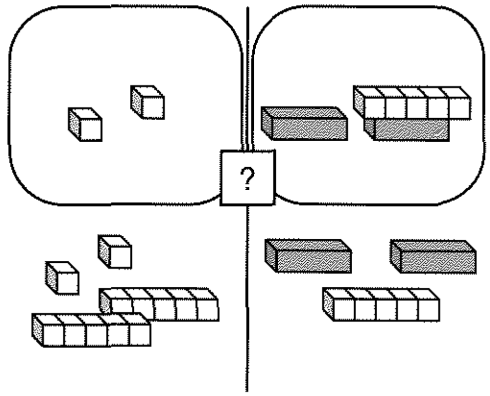
a. write the two expressions;
b. simplify both sides on the workmat;
c. decide which side is greater or whether they are equal, and write the correct sign between the expressions.

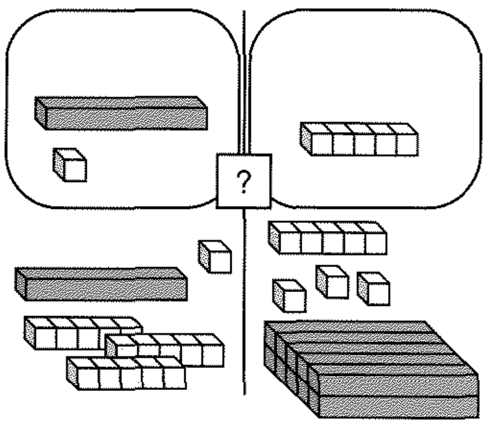
3.



4.

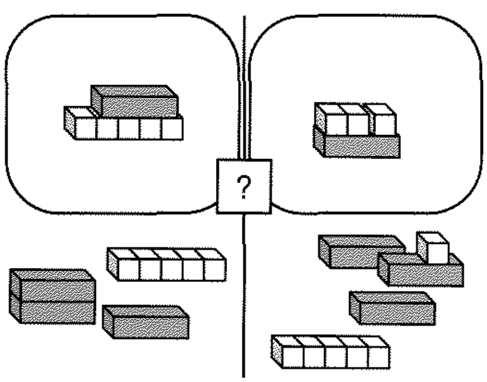


5. 

6. 

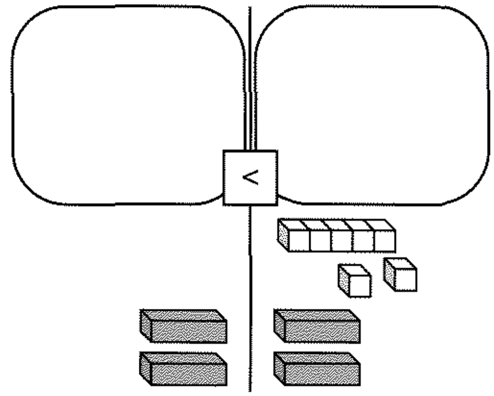
CAN YOU TELL?

To compare $2x - x + 5 - (5 - x)$ with $5 + 3x - 1 - (x - 3)$, first show the two expressions with the Lab Gear.



7. Simplify both sides, then arrange the blocks in a logical manner to determine which side is greater.

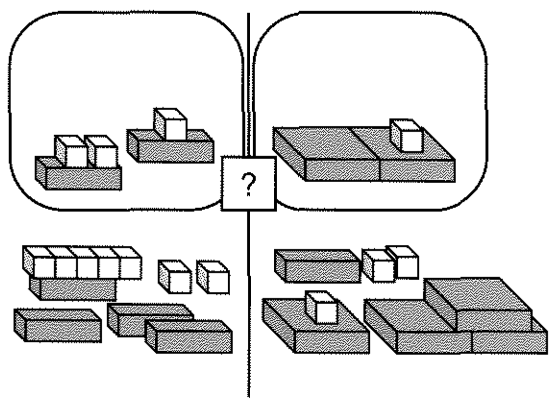
Your workmat should look like this.



Both sides include $2x$, but the right side is greater, as it also includes 7 more units. So we can write

$$2x < 2x + 7.$$

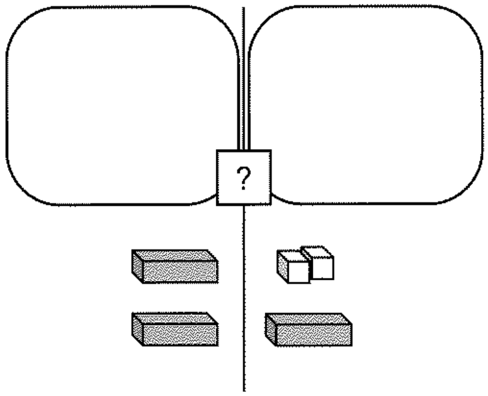
Now compare these expressions.



8. Write both expressions as they are shown in this figure.

9. Simplify both sides, then arrange the blocks in a logical manner to determine which side is greater.

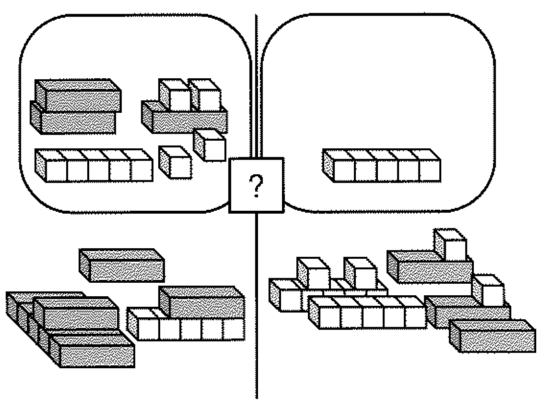
Your workmat should look like this.



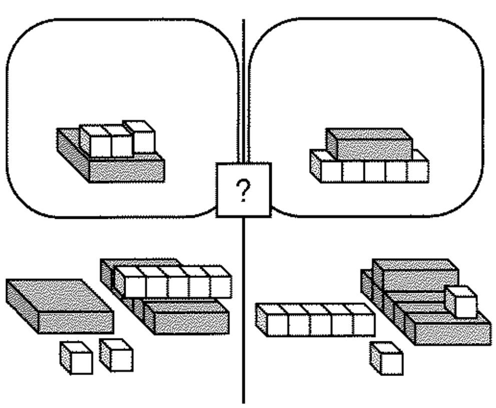
In this case, it is impossible to tell which side is greater, because we do not know whether x is greater or less than 2.

For problems 10-13, write both expressions as they are given. Then simplify, using your blocks, and write the expressions in simplified form. Decide which side is greater, whether they are equal, or whether it is impossible to tell. Write the correct symbol or ?.

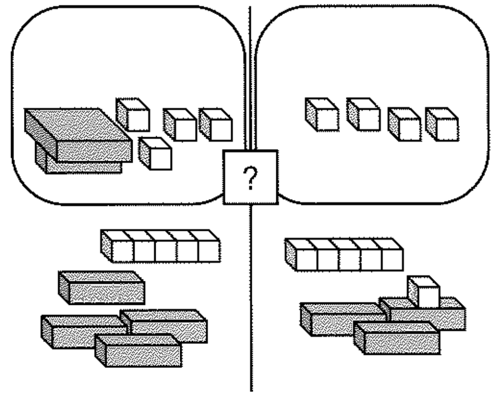
10.



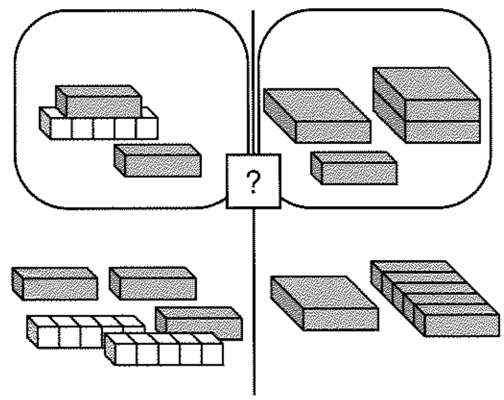
11.



12.



13.



TESTING VALUES OF X

Look at these two expressions.

$2x - 5$ $\quad\quad$ $-3x + 6$

Which is greater? The answer depends on the value of x.

14. a. Substitute -1 for x in both expressions and tell which is greater.

b. Substitute 3 for x in both expressions and tell which is greater.

c. Find another value for x which makes $2x - 5$ greater.

d. Find another value for x which makes $-3x + 6$ greater.

15. For each of the following pairs of expressions, find two values of x, one that makes the first expression greater and one that makes the second expression greater. Show all your calculations.

a. $7x - 4$ $\quad\quad$ $3x - 2$

b. $-2x + 6$ $\quad\quad$ $8x - 4$

c. x $\quad\quad$ $-x$

For each pair of expressions, write

A if the expression in column A is greater;
B if the expression in column B is greater;
? if you would have to know the value of x in order to know which is greater.

Remember that x can have negative and fractional values. It may help to think about the Lab Gear. In each case *explain your answer*, giving test values of x if it helps your explanation.

	A	B
16.	$7x$	$7x - 1$
17.	$7x + 1$	$-7x + 1$
18.	$7x + 1$	$7x - 1$
19.	$7x - 1$	$-7x - 1$
20.	$7x + 1$	$-7x - 1$
21.	$7x^2 - 1$	$7x - 1$

22. Compare your answers to problems 16-21 with other students' answers. Discuss your disagreements. If you disagree with another student, try to find an example to show which answer is not correct.

23. Write an expression containing x, that is less than 4 when x is less than 9.

24. Write an expression containing x, that is less than 4 when x is more than 9.

25. Write an expression containing x, that is less than 4 for all values of x.

DISCOVERY *MORE CODES*

If the coding function is of the form $y = mx$, it is more difficult to encode and decode. (For the letter values, see **Thinking/Writing 3.A**.)

26. a. Encode the word `extra` using $y = 3x$.
b. What did you do when $3x$ was larger than 26?

27. Decode the following sentence which was encoded with $y = 3x$. It may help to make a table showing the matching of the plaintext and coded alphabet.

`APIBOCEO HXO VOCIO.`

28. Encode the word `multiplication` with:
a. $y = x;$ b. $y = 2x;$
c. $y = 13x;$ d. $y = 26x.$

29.
a. Decode the following message, which was encoded with $y = 2x$. It may help to make a table showing the matching of the plaintext and coded alphabet.

`HD NPJ JRNPN NPRBN. DPN PDT FBB XDP NJXX TPBN'L JRNPN?`

b. What makes $y = 2x$ a difficult code to crack?

DISCOVERY *SUMMING UP*

Say that the sum of a word is the sum of the numbers corresponding to its letters. (For the letter values, see **Thinking/Writing 3.A**.) For example, the word `topic` has value

$$20 + 15 + 16 + 9 + 3 = 63.$$

30. What is the sum of the word `algebra`?

31. Find as many words as possible having sum 100.

LESSON 3.6

Multiplication and Division

You will need:

the Lab Gear

Notation: In algebra, the symbol ÷ is not used, perhaps because it looks too much like a + sign. To show division, use the format of a fraction.

$$\frac{6}{2} = 3$$

Or, if you're using a typewriter or computer, write it with a slash, $6/2 = 3$.

In this book we will write division both ways.

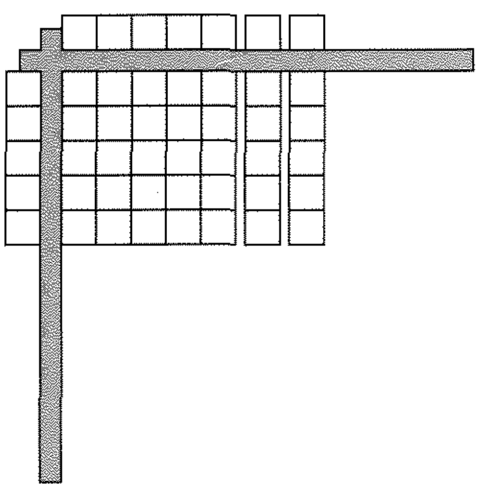
ONE MULTIPLICATION, TWO DIVISIONS

For most multiplication equations, there are two division equations. For example, corresponding to $7 \cdot 3 = 21$, we have

$21/7 = 3$ and $21/3 = 7$.

With the Lab Gear, you can use a rectangle to model multiplication and division.

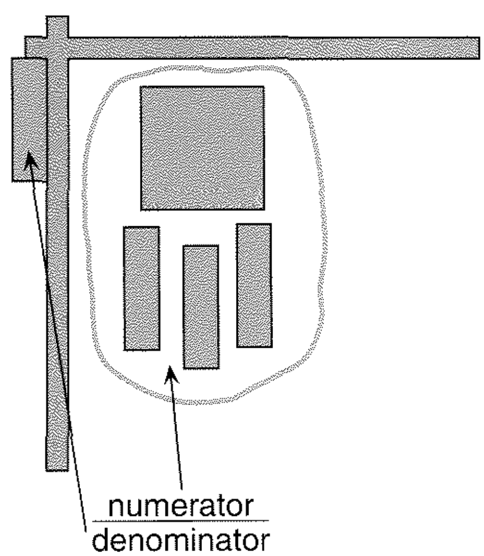
Arrange your corner piece and blocks to match this figure.



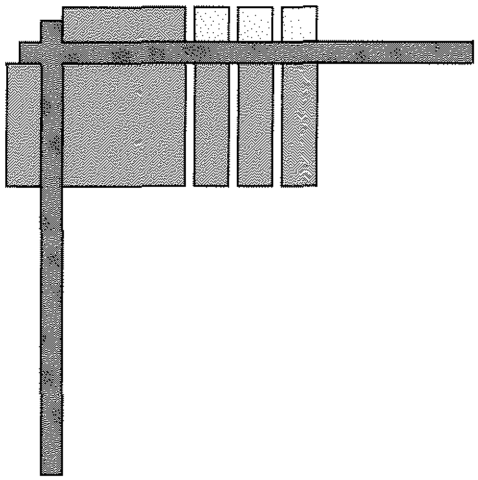
1. Write the multiplication equation that is shown by the figure.
2. Write the two division equations that are shown by the figure.
3. You could use the corner piece to set up several different divisions having numerator 12. For each, write the division equation and the corresponding multiplication equation.
4. Explain why it is impossible to set up the division 12/0 with the Lab Gear.
5. Some algebra students believe that $12/0 = 0$. Explain why they are *wrong* by discussing the multiplication that would correspond to this division.
6. a. Using the corner piece, multiply $(x + 4)(x + 3)$.
   b. Write two division equations related to the multiplication.

DIVISION IN THE CORNER PIECE

Here is an example of dividing in the corner piece.



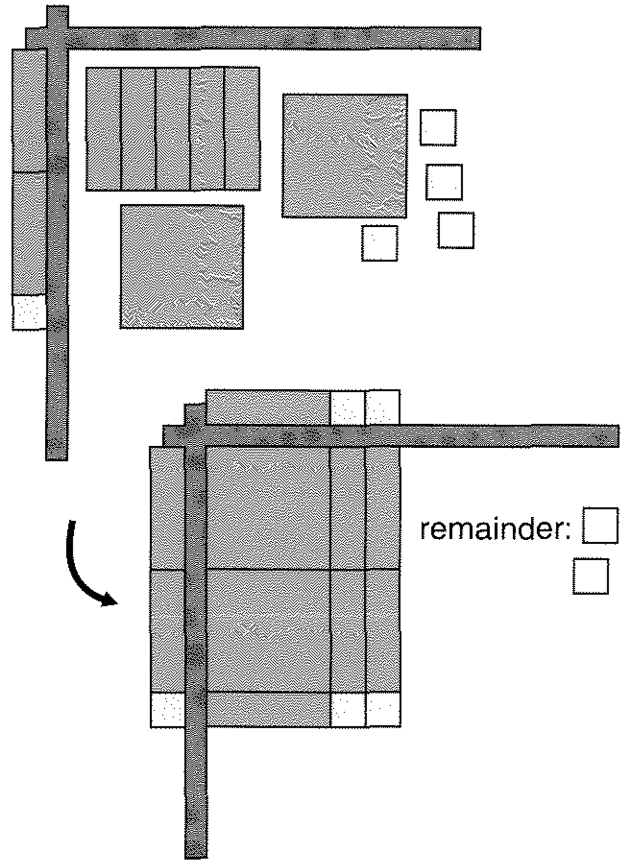
- Put the denominator to the left of the corner piece.
- Make a rectangle out of the numerator and place it inside the corner piece so that one side of the rectangle matches the denominator.



- Finally, to get the answer, figure out what blocks go along the top of the corner piece.

7. Write the division equation shown by the figure.

The denominator was a factor of the numerator, and a rectangle was formed with no pieces left over. However, in some cases, there will be a remainder. Here is an example.



8. What are the numerator, denominator, quotient, and remainder in the above division?

9. Divide.

a. $\frac{6x^2 + 3x}{3x}$ b. $\frac{9x + 3}{3}$

c. $\frac{x^2 + x + xy + y}{x + y}$ d. $\frac{xy + 2x + x^2}{x + y}$

e. $\frac{2x^2 + 6x + 4}{x + 2}$ f. $\frac{3x^2 + 10x + 5}{x + 3}$

10. For each division in problem 9, write the related multiplication equation.

11.

a. Divide. $\frac{y^2x + x^2y + 2xy + x^2 + y^2 + x + y}{x + 1}$

b. Write four multiplications having the product $y^2x + x^2y + 2xy + x^2 + y^2 + x + y$.

MULTIPLICATION WITHOUT THE LAB GEAR

Here is a method for multiplying polynomials without the Lab Gear. To perform the multiplication $(x + 2)(3y - 4x + 5)$, write the terms along the side and the top of a table.

	$3y$	$-4x$	5
x			
2			

Then enter the products of the terms in the corresponding boxes.

	$3y$	$-4x$	5
x	$3xy$	$-4x^2$	$5x$
2	$6y$	$-8x$	10

Then combine like terms, and you are done.

$(x + 2)(3y - 4x + 5) =$
$3xy - 4x^2 - 3x + 6y + 10$

Use this method for the following products.

12. $x(2x + 3xy + y^2)$

13. $(2x - y)(x + 3y)$

14. $(2x - y)(x - 3y)$

15. $(2x + y)(x - 3y)$

16. $(2x + y)(x + 3y)$

17. $(x + xy + 2yx)(y + 2 + x)$

This method for multiplication is a way to apply the distributive law to the multiplication of polynomials. *Every term must be multiplied by every term.*

MULTIPLICATION PUZZLES

Fill in the tables, including the polynomial factors along the side and the top. All coefficients are whole numbers. Is more than one solution possible for either table?

18.

$2x^3$		$-6x$
	$-x$	-3

19.

$12xy$	$15x^2y$	
		x^2y^2

20. Create a puzzle of this type that has a unique solution. Give the solver as few terms as possible.

21. Solve a classmate's puzzle.

REVIEW *WHAT'S YOUR SIGN?*

22. What is the sign of the missing factor?

a. $-123.4 \cdot$ ____ $= 567.89$

b. $98.76 \cdot (-54.3) \cdot$ ____ $= -21$

c. $98.76 \cdot (-54.3) \cdot$ ____ $= 0$

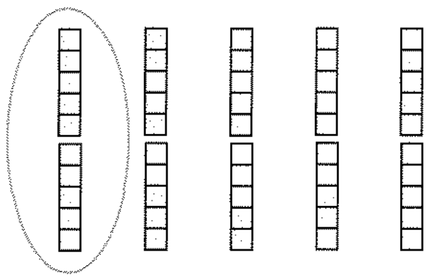
LESSON 3.7

Reciprocals

A MODEL FOR MULTIPLICATION

You cannot easily show multiplication by fractions with the Lab Gear, but the Lab Gear can help you think about it. For example, $(1/5) \cdot 50$ is read *one-fifth of fifty*. This means that we divide 50 into five parts and take one of them.

The diagram shows that $(1/5) \cdot 50 = 10$.



$(2/5)$ is two of five parts, so $2/5 \cdot 50 = 20$.

1. Find a number you could multiply by 8 to get a number less than 8.

2. Without finding its value, decide whether x would be more or less than 1. Explain how you know.

a. $8 \cdot x = 50$ b. $8 \cdot x = 5$
c. $8 \cdot x = 0.05$

3. Find the value of x for each equation in problem 2. (Hint: Remember that for any multiplication, there are two related divisions. You may use a calculator.)

A MULTIPLICATION SHORTCUT

4. Take 8, 3, and 2. They are three numbers whose product is 48. Another multiplication possibility is $6 \cdot 4 \cdot 2$. Find as many ways of writing 48 as a product of three different numbers as you can. Do not use 1 as a factor.

5. **Exploration** *Do not use 1 as a factor.*

a. Write 2 as a product of two different numbers.
b. Write 4 as a product of four different numbers.
c. Write 6 as a product of six different numbers.
d. Write 12 as a product of twelve different numbers.

Definition: The product of a number and its *reciprocal* is 1. Another way of saying this is, *the reciprocal of a number is the result of dividing 1 by the number.*

Examples: $3 \cdot 1/3 = 1$
$2/3 \cdot 3/2 = 1$
$0.31 \cdot 100/31 = 1$

6. Explain how the reciprocals of 3, 2/3, and 0.31 may have been found for the examples above. (No calculator was used.)

Guess the value of x, *without using your calculator.* If you think about reciprocals you will have to do very little arithmetic.

7. a. $5 \cdot \frac{1}{5} \cdot x = 6$
b. $4 \cdot x \cdot 9 \cdot \frac{1}{4} = 45$
c. $x \cdot 8 \cdot 7 = 8$
d. $x \cdot 8 \cdot 3 = 3$
e. $\frac{2}{3} \cdot x \cdot 3 \cdot \frac{1}{2} = 15$

8. a. $2 \cdot x \cdot 3 = 2$
b. $x \cdot 2 \cdot 2 \cdot 9 \cdot 3 = 6$
c. $\frac{1}{5} \cdot (5x) \cdot 3 = 1$
d. $\frac{1}{5} \cdot (5x) = \frac{3}{5}$

9. Make up two more equations like problems 7 and 8 and solve them.

10. Find two numbers a and b that will satisfy each equation. *Don't use your calculator.* Instead, think about reciprocals. Do not use 1 for a or b.

a. $a \cdot b \cdot 14 = 28$ b. $a \cdot b \cdot 28 = 14$

c. $\frac{2}{3} \cdot a \cdot b = 10$ d. $a \cdot b \cdot 10 = \frac{2}{3}$

RECIPROCALS ON THE CALCULATOR

Most scientific calculators have a key for reciprocals: $\boxed{1/x}$, or $\boxed{x^{-1}}$. (On calculators that do not have such a key, you can divide 1 by a number to find the number's reciprocal.)

11. Find the reciprocal of:

a. 1/23; b. 0.456; c. 7.89.

12. **Report** What is the result when you

a. multiply a number by its reciprocal?

b. divide a number by its reciprocal?

Be sure your results work for all numbers. Explain how you reached your conclusions.

13. $1/82 < 0.0123 < 1/81$. Explain.

14. Find two consecutive whole numbers such that 0.00123 is between their reciprocals.

15. Repeat problem 14 for 0.000123

16. Explain your method for solving problems 14 and 15.

A MODEL FOR DIVISION

17. **Exploration** Find a positive number such that when you divide that number by 5, your answer is

a. a number less than 1;

b. a number between 10 and 20;

c. a number greater than 100.

18. Find a positive number such that when you divide 5 by it, your answer is

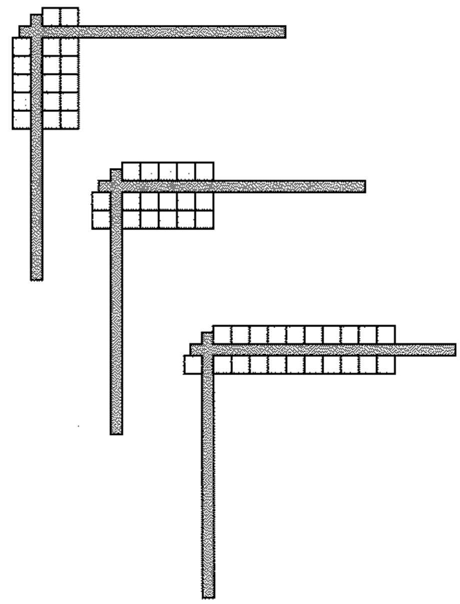
a. a number less than 1;

b. a number between 10 and 20;

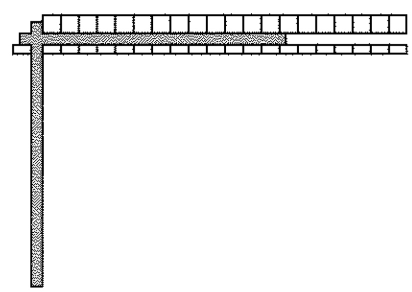
c. a number greater than 100.

Division by numbers between 0 and 1 is hard to show with the Lab Gear.

These diagrams show 10/5, 10/2, and 10/1.



What would 10/(1/2) look like? We cannot actually build this with the Lab Gear, but we could imagine what it would look like if we sliced each block in half.



19. a. What is the answer to the division shown in the figure?
    b. Dividing by 1/2 is equivalent to multiplying by what number?

20. a. Will the result of the division 8/(1/4) be more or less than 8?
    b. Use a sketch to show the division 8/(1/4).
    c. What is the answer to the division?
    d. Dividing by 1/4 is equivalent to multiplying by what number?

21.
    a. What is the result of the division of 8 by 0.1, 0.01, 0.001?
    b. What would happen if you divided 8 by a number that is much smaller than 0.001, almost equal to zero?
    c. How about dividing 8 by 0?

A DIVISION SHORTCUT

22. a. If you multiplied 5 by a number and got 30, what was the number?
    b. If you divided 5 by a number and got 30, what was the number?
    c. Compare your answers to parts (a) and (b). How are these numbers related?

23. *Dividing by a number is the same as multiplying by its reciprocal.* Explain, using examples.

Use this fact to perform each of the following divisions without your calculator.

24. 12/(1/4) **25.** 12/(2/3)

26. 10/0.4 **27.** $x^2/(1/x)$

SMALL NUMBERS

28. Find two numbers such that you get a result between 0 and 1 whether you add them, multiply them, subtract one from the other, or divide one by the other.

LESSON 3.8

A Hot Day

You will need:

graph paper

The sign at Algebank near Abe's house gives the time and temperature. The temperature is given two ways, using both the Celsius and Fahrenheit temperature scales. One hot day Abe made a record of the time and temperature at several times during the day. He tried to look at the bank sign exactly on the hour, but usually he was off by a few minutes. His data appear below.

Time	Temp (C)	Temp (F)
11:03	31	87
12:00	32	90
2:00	35	95
3:04	35	95
4:08	34	93
8:03	27	81

1. Exploration Abe heard on the radio that the low for the night had been 74 degrees (Fahrenheit) at 4:30 A.M. and the high for the day had been 97 degrees at 3:30 P.M. Using the information in the table, estimate what you think the Celsius readings on the bank sign would have been at those two times. Explain how you got your answers.

TEMPERATURE VARIATION

2. a. Draw a pair of axes on graph paper. Label the horizontal axis *Time* and the vertical axis *Temp*.
   b. Plot the points that show how the Celsius temperature changes with time. Your first point will be (11:03, 31).

3. a. Draw another pair of axes like the first one.
   b. Plot the points that show how the Fahrenheit temperature changes with time. Your first point will be (11:03, 87).

4. Write a short description of what your graphs show. Compare the two graphs.

COMPARING TEMPERATURE SCALES

A graph will help to show how the two temperature scales are related.

5. Draw a pair of axes. Put the Fahrenheit temperature on the vertical axis (label it *F*) and the Celsius temperature on the horizontal axis (label it *C*). Put the axes in the middle of your graph paper and leave plenty of room to extend your graph in all directions. Plot the points in Abe's table. Your first point will be (31, 87).

6. The points of your graph should fall approximately in a straight line. Draw a straight line that seems to go through most of the points.

Use your graph to estimate the answers to these questions. If necessary, extend your graph.

7. Approximately what is the
   a. Fahrenheit temperature when the Celsius temperature is 25°?
   b. Celsius temperature when the Fahrenheit temperature is 50°?
   c. Celsius temperature when the Fahrenheit temperature is -30°?

8. Is there a temperature where a Fahrenheit and Celsius thermometer show the same number? If so, what is it?

Abe's sister Bea wanted to estimate the Fahrenheit temperature for 17° Celsius. Someone had told her that the best way to remember the Celsius-Fahrenheit relationship was to memorize the fact that 16° Celsius is 61° Fahrenheit. Abe joked, "So 17° Celsius must be 71° Fahrenheit!" Bea replied, "I'll just add one degree. That means 17° Celsius must be 62° Fahrenheit."

9. Explain what Bea did wrong. Use your graph. Give examples explaining to Bea how to make the conversion correctly.

10. Judging from your graph, if you increase the Celsius temperature by one degree, by about how much does the temperature increase on the Fahrenheit scale?

CONVERTING CELSIUS TO FAHRENHEIT

Bea and Abe's parents, Mr. and Mrs. Gral, were planning a trip to Europe, where temperatures are given in Celsius. They asked their children to help them figure out how to convert from Celsius to Fahrenheit.

Abe asked his science teacher, who gave him the following rule: To get the Fahrenheit temperature, multiply the Celsius temperature by 1.8, then add 32.

11.
   a. Write a formula for this rule. Use F for the Fahrenheit temperature and C for the Celsius temperature.
   b. Check your formula by using it to convert one of the Celsius temperatures in Abe's table.

Bea looked up the subject in an almanac, which gave these instructions: To get the Fahrenheit temperature, multiply the Celsius temperature by 9, divide by 5, then add 32.

12.
   a. Write a formula for this rule.
   b. Check your formula by using it to convert one of the Celsius temperatures in Abe's table.

13. Compare the two formulas you wrote. Do you think they always give the same results? Explain, giving examples.

14. Use either method to convert these two Celsius temperatures to Fahrenheit.
   a. 20° Celsius = ___ Fahrenheit
   b. 21° Celsius = ___ Fahrenheit

15. According to your calculation in problem 14, when you increase the Celsius temperature by one degree, by about how much does the temperature increase on the Fahrenheit scale? Where does this number appear in the formula? Explain.

CONVERTING FAHRENHEIT TO CELSIUS

A journalist from Spain, G. Balear, is staying with the Grals. She is writing an article for a Spanish newspaper about her experiences in the United States. She wants to convert Fahrenheit temperatures to Celsius for her article.

16. The Fahrenheit temperature dropped to 41°. Bea is trying to help Ms. Balear convert it to Celsius. She has the idea of working backwards using the rule from the almanac. Use this method, or another method you think might work, to convert 41° F to Celsius.

17. Describe the method you devised in problem 16 for converting Fahrenheit to Celsius. Explain why it works. Show that it works for other temperatures by using it to convert some of the temperatures in Abe's table.

THINKING WRITING

3.B Opposites and Reciprocals

OPPOSITES

The function $y = -x$ can be thought of as the *opposite function,* since y and x are opposites.

1. a. Make a function diagram for the function $y = -x$.
b. Describe the in-out lines. (Are they parallel? Do they meet in a single point? If so, where is that point?)

2. To answer these questions, look at the diagram you made for problem 1.
a. As x increases, what happens to y?
b. Are x and y ever equal? Explain.
c. When x increases by 3, what happens to y?

3. Find the number and its opposite that are described. Use trial and error. Look for patterns. Try to develop a shortcut strategy.
a. a number 16 more than its opposite
b. a number 0.5 more than its opposite
c. a number 21 less than its opposite
d. a number A less than its opposite
e. a number 8 more than twice its opposite.

4. Report In a few paragraphs, summarize what you learned about opposites and their function diagrams. Include examples.

RECIPROCALS

The function $y = 1/x$ can be thought of as the *reciprocal function,* since y and x are reciprocals.

5. a. Make an in-out table for the function $y = 1/x$, using the following values for x: -5, -4, -3, -2, -1, -0.8, -0.6, -0.4, -0.2, and the opposites of these numbers (0.2, 0.4, etc.)
b. Make a whole-page function diagram for the function.

6. Use the function diagram you made in problem 5. Follow y with your finger as x goes up its number line. Answer these questions.
a. As x increases, what happens to y?
b. Are x and y ever equal?

7. On your function diagram of $y = 1/x$, as x moves up the number line, answer questions (a-h), describing what happens to y. (Does it move up or down? Fast or slowly? From what to what?)
a. when x is a negative number far from 0
b. when x approaches -1
c. when x passes -1
d. when x approaches 0
e. when x passes 0
f. when x approaches 1
g. when x passes 1
h. when x is a large positive number

8. Use your calculator to look for a number and its reciprocal that satisfy these requirements. If you cannot find an exact number, get as close as you can by trial and error. One is impossible.
a. The number is 9 times its reciprocal.
b. The number is 1/9 of its reciprocal.
c. The number equals the opposite of its reciprocal.
d. The number is 3 times its reciprocal.
e. The number is one more than its reciprocal.

9. Report Summarize what you learned about reciprocals and their function diagrams. Include examples. (Do not forget to discuss what happens when $x = 0$.)

Equations and the Cover-Up Method

WRITING EQUATIONS

A seamstress makes dresses for a living. After an illness, she has only \$100 in her business bank account. She takes out a \$1000 loan at Algebank. The interest on the loan is \$15 per month if it gets paid back in the first year. She spends \$720 on dress-making materials, and keeps the rest in her bank account to cover additional costs, such as sewing machine repairs or whatever else may come up. Materials for one dress come to \$20. She makes two dresses a day, four days a week, and spends one day a week selling the dresses and dealing with other matters related to her business.

She sells as many dresses as she can to private customers for \$160 each, and the rest of the dresses to stores, for \$100 each. She needs \$750 a week for living expenses and puts any income over that in her bank account. She hopes to pay back her loan, and to make enough money so that when she needs to buy more materials, she does not have to take out another loan. Can the seamstress meet her goals? How could she improve her financial situation?

One way to think about a problem like this one is to break it down into smaller problems, and to write and solve equations for those. For example, let's write an expression that would tell us how much money the seamstress puts in her bank account every week.

1. **Exploration** Assume the seamstress has x private customers a week. Answer the following questions for one week, in terms of x.
   a. How many dresses does she sell to stores?
   b. How much money does she receive from private customers?
   c. How much money does she receive from stores?
   d. What is the total amount of money she receives every week?
   e. How much of it is she able to put in her bank account? Simplify your answer.

If you answered the questions correctly, you should have ended up with the expression $60x + 50$ for the amount she deposits every week as a function of x. Let's say that she would like this amount to be \$300. This gives us the equation $60x + 50 = 300$. Remember that x is the number of private customers per week. We can now find out how many private customers she would need to deposit \$300 per week. All we need to do is solve the equation.

SOLVING EQUATIONS

Definition: Finding all the values of a variable that make an equation true is called *solving the equation.*

You have already solved equations by trial and error. The *cover-up method* is another technique for solving equations. It is based on the idea of working backwards.

Example 1: $60x + 50 = 300$

With your finger, cover up the term that has the x in it. The equation looks like

$$\square + 50 = 300.$$

Clearly, what's in the box is 250. So

$$60x = 250.$$

Think of a division that is related to this multiplication, and you will see that

$$x = 250/60$$

or $$\mathbf{x = 4.1666....}$$

So in order to deposit \$300 a week, the seamstress needs to have more than four private customers a week.

Example 2: This one is about a more complicated equation.

$$5 + \frac{3x - 1}{4} = 7$$

Cover up the expression $\frac{3x - 1}{4}$. You get

$$5 + \square = 7.$$

Whatever is hidden must be equal to 2. So

$$\frac{3x - 1}{4} = 2$$

Now cover up $3x - 1$ with your finger.

$$\frac{\square}{4} = 2$$

What is under your finger must be 8. So

$$3x - 1 = 8.$$

Cover up the term containing x.

$$\square - 1 = 8$$

What's under your finger must equal 9. So

$$3x = 9$$

and $\boldsymbol{x = 3}$.

2. Check the solutions to examples 1 and 2 by substituting them in the original equations.

Solve each equation. Use the cover-up method, then check each answer by substituting.

3. a. $3(x - 10) = 15$
b. $3(x + 10) = 15$
c. $3 + \frac{x}{10} = 15$
d. $\frac{18}{x} + 12 = 15$

4. a. $34 - \frac{2x + 6}{2} = 4$
b. $34 - \frac{2x + 6}{2} = -4$

5. a. $21 = 12 + \frac{3x}{8}$ b. $12 = 21 + \frac{3x}{8}$

6. a. $5 + \frac{x}{6} = 17$ b. $5 + \frac{6}{x} = 17$
c. $5 - \frac{x}{6} = 17$ d. $5 - \frac{6}{x} = 17$

7. a. $3 = \frac{12}{x + 1}$ b. $3 = \frac{x + 1}{12}$
c. $3 = \frac{12}{x + 7}$ d. $3 = \frac{x + 7}{12}$

8. Make up an equation like the ones above that has as its solution
a. 4; b. -4; c. $\frac{1}{4}$.

Since the cover-up method is based on covering up the part of the equation that includes an x, it can be used only in equations like the ones above, where x appears only once. In other equations, for example

$$160x + 100(8 - x) - 750 = 300,$$

you cannot use the cover-up method, unless you simplify first.

9. Find out how many private customers the seamstress needs every week so that, at the end of four weeks, she has enough money in her bank account to pay back her loan and buy dress-making materials for the next four weeks. Use equations and the cover-up method if you can. Otherwise, use any other method. In either case, explain how you arrive at your answers.

REVIEW DIVIDING BY ZERO

10. Explain, using multiplication, why $20/5 = 4$.

11. Explain, using multiplication, why 20/0 is not defined. (Hint: Start by writing $20/0 = q$. Write a related multiplication. What must q be?)

12. Explain, using multiplication, why 0/0 is not defined. (Hint: Start by writing $0/0 = q$. Write a related multiplication. What must q be? Could it be something else?)

DISCOVERY BE PRODUCTIVE

Say that the product of a word is the product of the numbers corresponding to its letters. (For the letter values, see **Thinking/Writing 3.A**.) For example, the word `optic` has value

$$15 \cdot 16 \cdot 20 \cdot 9 \cdot 3 = 129{,}600$$

13. What is the product of the word `ALGEBRA`?

14. Find words whose product is as close to one million as possible.

15. Find words having these products. (Hint: It would help to find the prime factors of the numbers.)

a. 6	b. 8
c. 12	d. 14
e. 15	f. 16
g. 20	h. 24
i. 35	j. 455
k. 715	l. 2185
m. 106,029	n. 4,410,000

LESSON 3.10

Combining Functions

You will need: graph paper

DIAGRAMS OF COMBINED FUNCTIONS

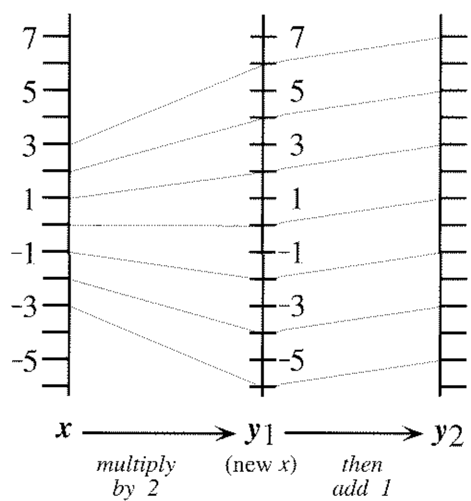
Function diagrams can be used to show the result of combining functions. Here are two simple functions. One function doubles x. The other function adds 1 to x.

$y_1 = 2x \qquad\qquad y_2 = x + 1$

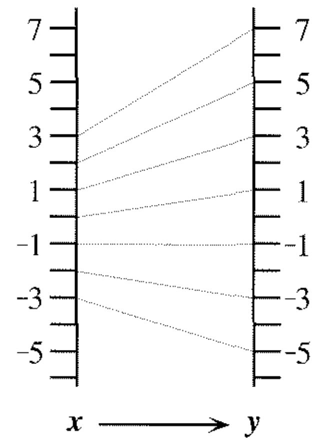
Notation: The 2 in the name y_2 is called a *subscript.* It is written lower and smaller than the y. It does *not* mean *multiply by 2* or *square.* It is just a way to distinguish two variables that would otherwise have the same name.

1. Draw function diagrams for y_1 and y_2.

This two-step function diagram shows one way of combining y_1 and y_2. First, double x. Then add 1 to the result. The y value of y_1 becomes the new x value for y_2.



These two steps can be combined as shown in this one-step function diagram.



2. Write a rule for this function diagram.

The functions y_1 and y_2 can also be combined in the other order: First, add 1 to x. Then double the result. The y value of y_2 becomes the new x value for y_1.

3. Draw a two-step function diagram showing the combination of the functions in this order.

4. Summarize your two-step function diagram in a one-step function diagram.

5. Write a rule for the one-step function diagram you drew.

6. Does the order in which we combine the functions matter? Explain.

These problems are about the following two functions.

$y_1 = -3x \qquad\qquad y_2 = x + 2$

7. Show a two-step function diagram, combining the functions by performing y_1 first and then y_2.

8. Summarize your two-step diagram in a one-step diagram and write the function that corresponds to your one-step function diagram.

9. Repeat problems 7 and 8, but this time combine the two functions by performing y_2 first, followed by y_1.

10. Did the resulting function change, when you changed the order in which you combined the two functions? Explain.

11. Exploration Sometimes you can combine two functions in either order and the resulting function is the same. Find pairs of functions that have this property. You may use function diagrams to verify your answer. Discuss any patterns you notice.

INVERSE ACTIONS

The inverse of an action is the action that undoes it. For example, suppose you were leaving home in the car. You would perform these four actions.

ACTION 1: Open the car door.
ACTION 2: Get into the car.
ACTION 3: Close the door.
ACTION 4: Start the car.

If, before driving away, you suddenly realized that you forgot something, you would have to undo all these actions. You would undo the actions in the reverse order:

First, UNDO ACTION 4: Stop the car.
Second, UNDO ACTION 3: Open the door.
Next, UNDO ACTION 2: Get out of the car.
Last, UNDO ACTION 1: Close the car door.

12. Describe how to undo these actions.

a. In the morning, you put on your socks, then put on your shoes. What do you do in the evening?

b. To take a break from this homework, you close your math book, stand up from your desk, turn on the television, and sit down on the sofa. What do you do to get back to work?

13. Al believes that the way to undo the actions *open the car window; stick your head out* is *close the car window; pull your head in.* Comment on this idea.

14. Create your own example of inverse actions.

INVERSE FUNCTIONS

15. Exploration Choose any function and make a function diagram for it. Then draw the mirror image of this function diagram. What is the function associated with the mirror image? How is it related to the original function? Try this with several functions. Write about any patterns you notice.

The inverse of a function is a function that undoes it. For example, look at these two input-output tables.

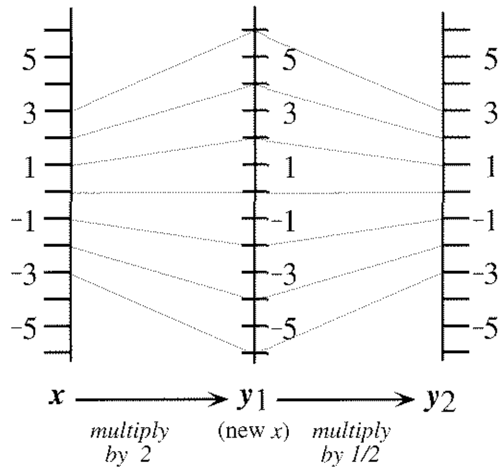
x	y
2	6
-1	-3
4	12

x	y
6	2
-3	-1
12	4

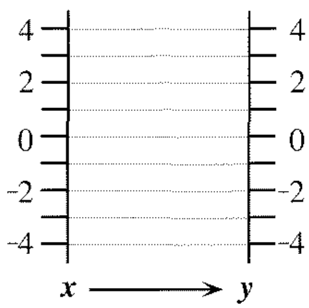
16. a. What happens when you use an output from the first table as the input for the second table?

b. What two functions do you think are represented by these two tables? How are the functions related?

If $y_1 = 2x$ and $y_2 = (1/2)x$, a two-step function diagram shows that y_2 undoes y_1.



This is shown dramatically when the two-step diagram is summarized in a one-step diagram.



A function y_1 performs the following operations on a number.

Multiply the number by 3, subtract 1.

17. Write in words what the inverse function does. (Call it y_2.)

18. a. Write a rule in the form $y_1 =$ for the original function.

b. Write a rule in the form $y_2 =$ for the inverse function.

19. a. Make separate function diagrams for y_1 and y_2.

b. Describe how the diagrams you made are related.

20. Make a two-step function diagram for the combination of y_1 and y_2.

21. Make a one-step function diagram summarizing your two-step diagram. Would it matter if you combined y_1 and y_2 in the other order?

22. Summary Write a summary of what you have learned in this lesson about combining function diagrams, especially those of inverse functions. Use examples.

23. Find functions that are their own inverses. What do you notice about their function diagrams? Explain.

LESSON 3.11

Math on Another Planet

SMALL POCKETS

On the treeless planet of Glosia, the currency consists of florins, ecus, and ducats. *One florin is worth two ecus, and one ecu is worth two ducats.* Since there is no paper, there is no paper money, and the people of Glosia have to carry coins everywhere. King Evariste VII, being immensely rich, must wear bloomers with enormous reinforced pockets to hold his money.

One day the King realizes that there is a new trend in Glosian fashion. Elegant men and women wear only small pockets. Evariste VII, not one to be left behind by the great movements of style, decides to institute a drastic economic reform by enacting a strange law: *One ducat is worth two florins!* (The old rules are not changed.) When you realize trades can be made in either direction, you can see how the King's brilliant legislation will abolish poverty forever.

The people of Glosia are ecstatic. With the new system, one may have a fortune in one's pockets, and yet never carry more than three coins! One can be rich and fashionable at the same time. For example, if you own eight ecus, you can go to the bank, and trade them in for four florins. These can be traded again, for two ducats, which equal one ecu, which will certainly fit in your pocket.

1. Exploration

a. The King trades his coins at the bank, according to their official value, with the object of having as few coins as possible in the tiny pocket of his slinky new pants. He starts with 1000 florins. What does he end up with?

b. Prince Enbel has one ducat. He buys a toastereo (a popular appliance which, unfortunately, does not make coffee), costing 50 ecus. If he is given the fewest coins possible, how much change does he get?

c. Princess Lisa has one ecu. She wins the first prize in a contest in *Names* Magazine. The prize is one ducat, one ecu, and one florin. She now has four coins, but they won't fit into her pocket. What does she have after trading them in to get as few coins as possible? (The second prize would have been a T-shirt with the *Names* logo and no pockets at all.)

d. Sol Grundy has no money. He gets a job at the toastereo store, earning one florin per day, seven days a week. Since his pockets are fashionably small, he trades his money as often as possible in order to have as few coins as possible. If he starts his new job on Monday, how much does he have each day of the week? The next week? (Assume he doesn't spend any money.)

2. Make a list of the amounts of money one can have that cannot be reduced to a smaller number of coins. (Hint: There are seven possible amounts.) One of the amounts is $(d + e)$.

3. Make an addition table for Glosian money. It should be a seven-by-seven table, with a row and column for each of the amounts you found in problem 2. For example, your table should show that $(d + e) + d = f$.

4. One of the seven amounts you found in problem 3 can be considered to be the "zero" of Glosian money, since adding it to a collection of coins does not change the collection's value (after trading to get the smallest possible number of coins). Which amount is the zero for Glosian money?

5. The opposite of an amount is the amount you add to it to get the zero. Find the opposite of each of the seven amounts in problem 3.

A LONG MONTH

The King can never remember which month it is and how many days the month has. He decides to start a new calendar, with a single infinite month, the month of *Evary,* named after himself. This is what the calendar looks like.

Evary						
Mo	**Tu**	**We**	**Th**	**Fr**	**Sa**	**Su**
			1	2	3	4
5	6	7	8	9	10	11
12	13	14	15	16	17	18
19	20	21	22	23	24	25
26	27	28	29	30	31	32
33	34	35	36	37	38	...

6. What day of the week will it be on Evary 100th? Explain how you figured it out.

The King is so pleased with the new calendar that he decides to invent a new kind of math. He calls it *Calendar Math.* In Calendar Math, Monday + Tuesday →

$$5 + 6 = 11 \rightarrow \text{Sunday},$$

or, more briefly, Mo + Tu = Su.

7. Check whether, if you picked different numbers for Monday (such as 12, 19, etc.) and Tuesday (13, 20, etc.), you would still get Sunday for the sum.

8. Make an addition table for Calendar Math. It should be a seven-by-seven table, with the days of the week along the left side and across the top and their sums inside the table.

9. *Calendar Zero* is a day of the week such that, when you add it to any other day, you get that other day for the answer. What day is Calendar Zero?

10. Find the *Calendar Opposite* for each day of the week. That is the day you add to a given day to get Calendar Zero. If a day does not have an opposite, or is its own opposite, explain.

11. Calculate.

a. Mo + Mo

b. Mo + Mo + Mo

c. Mo + Mo + Mo + Mo, etc.

12. How many times do you add Mo to itself to get back Mo?

13. Make a multiplication table for Calendar Math. Here is an example of a result that would appear in it.

Mo · Tu → 5 · 6 = 30 → Fr,

so, Mo · Tu = Fr.

14. What is special about Calendar Zero in multiplication?

15. *Calendar One* is a day of the week such that when you multiply it by any other day, you get that other day for the answer. What day is Calendar One?

16. The *Calendar Reciprocal* of a day is the day you multiply it by to get Calendar One. Find the Calendar Reciprocal for each day. If a day does not have a reciprocal, or is its own reciprocal, explain.

17. Calculate Su^2, Su^3, etc. What power of Su is equal to Su?

18. Summary Summarize Calendar Math.

LESSON 3.12

Similar Figures

You will need:

- geoboards
- dot paper

EQUIVALENT FRACTIONS

1. Using a rubber band, connect the origin and (6, 9). The line misses most geoboard pegs, but it goes *exactly* over two of them (in addition to the pegs it connects). What are their coordinates?

Problem 1 provides a way to find equivalent fractions on the geoboard. If you think of (6, 9) as representing 6/9, you have found two other fractions equivalent to it, making this a set of three equivalent geoboard fractions.

2. **Exploration** Find as many sets of equivalent geoboard fractions as possible. Do not use zero in the numerator or denominator. There are 56 fractions distributed in 19 sets. Do not include sets that consist of just one fraction.

ENLARGING WITHOUT DISTORTION

3. a. Make the face of an alien with rubber bands on your geoboard. The whole face needs to fit in the bottom left quarter of the board. In other words, none of the coordinates can be greater than 5. Don't make it too complicated.

b. Make a record of the coordinates you used. You will need those in the next problems.

c. Copy the face on dot paper.

4. Doubling the x-coordinates and leaving the y-coordinates the same, make a copy of your alien's face on dot paper. This is called the $(2x, y)$ copy.

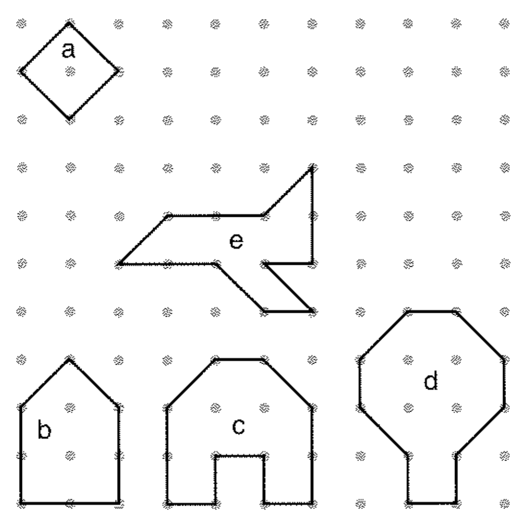
5. Repeat problem 4, but this time leave the x-coordinates as in the original and double the y-coordinates only. This is called the $(x, 2y)$ copy.

6. Repeat problem 4 again, with both x- and y-coordinates doubled. This is called the $(2x, 2y)$ copy.

7. **Summary** Write a paragraph answering these questions: Which of the copies looks most like an enlarged version of the original? How are the other copies distorted?

8. Write a story about the alien's adventures, explaining why its face went through these changes.

9. Enlarge the following figures without distortion. Explain how you did it.



SIMILAR RECTANGLES

Definition: When one figure can be obtained from another by enlarging it or shrinking it without distortion, the figures are said to be *similar.*

10. Make a rectangle having vertices at (0, 0), (4, 0), (4, 6), and (0, 6). Find a smaller rectangle that is similar to it by finding a number you can multiply the given coordinates by to get whole number coordinates that will fit on the geoboard. Sketch both on the same figure.

11. Repeat problem 10, but find a larger rectangle that is similar to the given one. Sketch it on the same figure as in problem 10.

The following questions are about the three rectangles from problems 10 and 11.

12. Connect the origin with the opposite vertex in the largest rectangle. Does your rubber band pass through vertices of the other two rectangles?

13. What are the length and width of each rectangle? How are they related to each other?

14. Can you think of a *single number* that tells what all three rectangles have in common?

Here are two ways to tell whether two rectangles are similar.

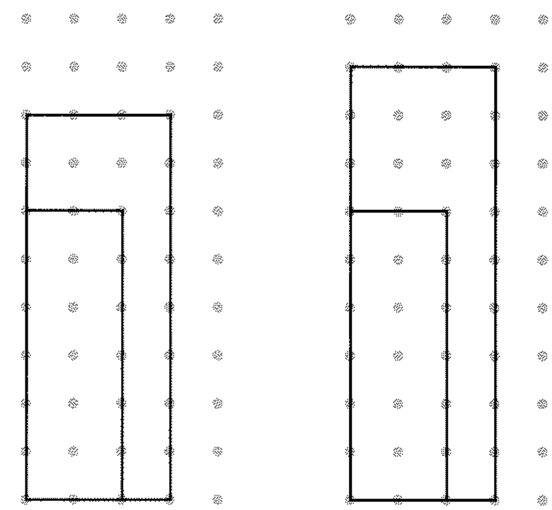
Geoboard diagonal method: Make both rectangles in the bottom left of a geoboard, with one vertex on the origin, and sides along the x- and y-axes. Then connect the origin to the opposite vertex of the larger rectangle. If the diagonal you created passes exactly over the vertex of the smaller rectangle, they are similar.

Calculator division method: Check whether the ratio of the dimensions is the same in both the rectangles.

Example:

a. a 2-by-6 rectangle and a 3-by-8 rectangle

b. a 2-by-6 rectangle and a 3-by-9 rectangle



$2/6 = 0.3333333...$

$3/8 = 0.375$

$3/9 = 0.3333333...$

15. Explain the results of the two methods in this example.

You may know other methods for recognizing whether fractions are equivalent. You can use those also, to determine whether rectangles are similar.

16. Summary Explain how the ideas of *similar rectangles* and *equivalent fractions* are related.

17. Are these two rectangles similar? The first one has vertices: (0, 1), (2, 0), (4, 4), and (2, 5). The other one has vertices (7, 3), (9, 6), (3, 10), and (1, 7). Since the methods outlined above will probably not work, explain how you arrive at your answer.

REVIEW THE COMMUTATIVE AND ASSOCIATIVE LAWS

18. Write an expression using
- the numbers 1, 2, and -3, in any order,
- two subtractions,

in as many ways as possible.

In each case, calculate the value of the expression.

Examples: $2 - 1 - {-3} = 4$
$2 - (1 - {-3}) = {-2}$
$({-3} - 1) - 2 = {-6}$

19. Do the commutative and associative laws apply to subtraction? Explain.

DISCOVERY CLOCKMATH

Clock Math can be defined by saying that only the numbers on the face of a clock (1, 2, …, 12) are used. In Clock Math, $5 + 9 = 2$, and $5 \times 9 = 9$. This is because when you pass 12, you keep counting around the clock.

20. Report Write a report on Clock Math. You may start with a science fiction or fantasy story to explain an imaginary origin for Clock Math. Your report should include, but not be limited to, answers to the following questions: Is there a *Clock Zero*? What is it? Does every number have a *Clock Opposite*? What is it? Is there a *Clock One*? Does every number have a *Clock Reciprocal*? What is it? Don't forget to make addition and multiplication tables.

THINKING WRITING 3.C More Banking

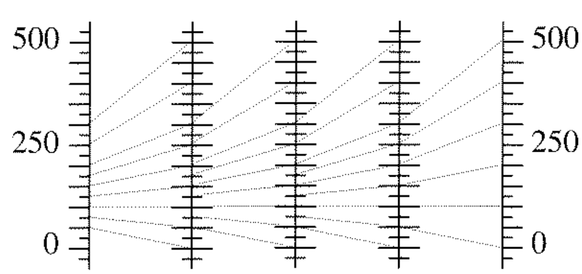
Reg works for Algebank. He was trying to analyze the investment plan described in the first lesson of this chapter. He decided to use x's and y's in his analysis. He wrote:

x = amount of money the person invests
y = amount of money the person has after one month

Since the bank doubles the investor's money and deducts the \$100 fee, the function relating x and y is $y = 2x - 100$.

1. Make a function diagram for this function.
2. Use your function diagram to find out
   a. how much an investor, who had \$300 after one month, started with;
   b. how much an investor, who started with \$300, had after one month.
3. Use your function diagram to find the amount of money the investor started with, who ended up with the same amount of money after one month. (This is called the *fixed point* of the function.)
4. What happens to an investor who starts out with an amount of money less than the fixed point? With an amount of money greater than the fixed point?

To analyze what happens to an investment over a period of more than one month, Reg connected function diagrams. Since the amount at the end of the first month is the amount at the beginning of the second month, he used the y-number line from the first diagram as the x-number line of the next, doing this many times.



5. Describe what the linked function diagrams show.
6. How could one use a single-function diagram to follow what would happen to an investment over a period of more than one month?
7. Use Reg's method to analyze a plan where the investment is multiplied by 1.5 and the service charge is \$50. Describe what your linked diagrams show.
8. Compare the plan in problem 7 with the first plan for someone who invests
   a. \$90; b. \$100; c. \$110.
9. Which do you think has a bigger influence on the amount of money the investor makes, the service charge, or the number by which the investment is multiplied? Write an explanation supporting your opinion. Use several examples.
10. Explain why Al thought it was important to know whether the service charge was deducted before or after the money was doubled. Use some examples. Express each policy with a function.
11. **Report** Write a report on investment plans of the type studied in this assignment and in Lesson 1, plus, optionally, other plans of your design. Use variables. Your report should include, but not be limited to, answers to problems 9 and 10.
12. **Project** Find out what the service charge and interest rate are at three real banks. Figure out what would happen to \$100 invested at each service charge and interest rate over a period of three years. Write up what you discover as if it were an article for the school newspaper, and you were giving advice to students.

Essential Ideas

WORKING BACKWARDS

Abe and Bea had baked a batch of cookies. They told Reg, Al, and Lara that they could each have one-third of the cookies. Later, Reg went into the kitchen and took one-third of the cookies. An hour after that, not knowing that Reg had already taken his share, Lara claimed one-third of the remaining cookies. A few minutes later Al, thinking he was the first to find the cookies, devoured one-third of what was left.

1. If 8 cookies are left, how many must Abe and Bea have baked?

TWO NEGATIVES

2. Find the sign of the result.

a. $3 - 5$ b. $3 - (-5)$
c. $-5 - (3)$ d. $-5 - (-3)$

3. Find the sign of the result.

a. $-(5)(-3)$ b. $-(5 - 3)$
c. $-[-3 - (-5)]$ d. $-(-5)(-3)$

POSITIVE, NEGATIVE, OR ZERO?

4. For each expression, write *P, N,* and/or 0, depending on whether it can possibly be positive, negative, or 0. (Try various values for the variables to help you decide. For example, -2, 0, and 2.) Explain your answers.

a. $5x$ b. $-2x^2$
c. $-9y$ d. $5y^2$
e. z^3 f. $-a^4$

SIMPLIFYING EXPRESSIONS

Simplify each expression.

5. $12x - 6xy - (-3x) - (-2y)$

6. $-3x^2 - (3)2 + x^2 - (2 - x^2)$

7. $x - (x - 5) - (5 - x)$

FROM WORDS TO ALGEBRA

8. a. Translate each step into algebra.
1) Think of a number.
2) Add 4.
3) Multiply the result by 2.

b. If I got 46, what was my original number?

9. a. Translate each step into algebra.
1) Think of a number.
2) Multiply by 2.
3) Add 4.

b. If I got 46, what was my original number?

c. Compare your answer to part (b) with your answer to part (b) in problem 8. Were your answers the same or different? Explain.

COMPARING EXPRESSIONS

10. Find a value of x for which

a. $-8x - 1$ is less than $8x + 3$;
b. $-8x - 1$ is greater than $8x + 3$;
c. $-8x - 1$ is equal to $8x + 3$.

MULTIPLICATION TABLES

Find these products. Combine like terms.

11. $(x + 3)(2x + 4)$

12. $(x + 3)(2x + 4y)$

13. $(x + 3 + y)(2x + 4y)$

Fill in the blanks.

14.

	x	-3	$5y$
____	$2x^2$	$-6x$	$10xy$

15.

	____	____	____
$3y$	$-6x^2y$	$15y^3$	$-3y$

16. ____ $(x - 2) = 2 - x$

OPPOSITES AND RECIPROCALS

17. Simplify each expression. Look for shortcuts.

a. $9 \cdot \frac{1}{3} \cdot \frac{2}{3} \cdot 5 \cdot \frac{3}{2}$

b. $[5x - (\text{-}5x)] - [5x - (\text{-}5x)] - 16x$

c. $0.5 \cdot 25 \cdot 0.02 \cdot 2$

18. Gabe and Abe were arguing about xy. Gabe said that the opposite of xy is yx. Abe said that the opposite of xy is $\text{-}xy$. Lara overheard them, and said she thought that the opposite of xy is $\text{-}yx$. Write an explanation that will settle their argument.

19. What numbers are

a. greater than their reciprocal?

b. less than their reciprocal?

c. equal to their reciprocal?

d. less than their opposite?

e. equal to their opposite?

20. a. Which of the following is the reciprocal of $3x$?

$$\frac{1}{3x}, \frac{3}{x}, \text{ or } \frac{1}{3}$$

b. Check your answer by substituting two different numbers for x and showing that the product of $3x$ and its reciprocal is 1 in both cases.

INVERSE FUNCTIONS

Write the inverse of each of the following functions.

21. a. The function adds 2 to x and multiplies the result by 4.

b. The function multiplies x by 4 and adds 2 to the result.

c. $y = 7x - 4$

22. a. The function takes the opposite of x.

b. The function takes the opposite of x, adds 5, and divides the result by 2.

c. $y = \frac{3 - x}{6}$

Scientists sometimes use the Kelvin temperature scale. To convert Kelvin temperatures to Celsius, you subtract 273. For example, the melting temperature of iron is 1808° Kelvin, or 1535° Celsius.

23. Lead melts at 600° Kelvin. What temperature is that in Fahrenheit? (Use the information from Lesson 8.)

24. Explain how to convert Kelvin temperatures to Fahrenheit, and how to convert Fahrenheit to Kelvin. (Hint: Use arrows to show each step of the conversion.)

25. a. Make a function diagram for the function $y_1 = (x/2) + 1$.

b. Make the function diagram of its inverse and find the rule.

c. Find the function that results from combining y_1 and its inverse. Does the order in which you combine the functions matter? Explain.

SOLVING EQUATIONS

Use the cover-up method to solve these equations.

26. $\frac{24}{x - 5} + 3 = 9$ **27.** $\frac{x - 5}{24} + 3 = 9$

28. $\frac{5 - x}{24} + 3 = 9$ **29.** $\frac{24}{5 - x} + 3 = 9$

30. Compare the solutions to each pair of equations. (Use related multiplication equations.)

a. $\frac{2}{M} = 6$ and $\frac{6}{M} = 2$

b. $\frac{8}{M} = 4$ and $\frac{4}{M} = 8$

c. $\frac{20}{M} = 5$ and $\frac{5}{M} = 20$

d. Make up another example like this.

31. Describe the pattern you found in problem 30. Explain why it works.

PRACTICE

REVIEW/PREVIEW SIMPLIFY

1. $x + 0.2x$

2. $x - 0.2x$

3. $x + 0.8x$

4. $x + (1/4)x$

5. $x - (1/4)x$

PREVIEW EQUAL RATIOS

The equations below all involve two equal ratios. Find the value of x that will make the ratios equal. You may want to use trial and error with your calculator.

6. $\frac{x}{4} = \frac{6}{1}$

7. $\frac{3}{x} = \frac{5}{7}$

8. $\frac{x}{3} = \frac{5}{7}$

9. $\frac{3}{1} = \frac{6}{x+7}$

10. $\frac{4}{5} = \frac{6}{x+7}$

REVIEW/PREVIEW EQUATIONS

11. For each equation, use trial and error to find a value of n that makes it true.

a. $3n + 10 = 5n$

b. $5n + 10 = 3n$

c. $7n + 10 = 8n$

d. $8n + 10 = 7n$

12. Use trial and error or the cover-up method to solve these equations.

a. $2(x + 5) = 8$

b. $5 + 2(x + 4) = 19$

c. $3(2x + 4) - 7 = 11$

d. $-4(10x - 3) - 6 = -14$

13. Find a positive integer that satisfies each equation.

a. $3n - 1 = 47$

b. $n^2 - 5 = 59$

14. Find a negative integer and a positive integer that satisfy the equation

$$n^2 - n = 20.$$

CHAPTER 4

The spiral curve of a West African chameleon's tail

Coming in this chapter:

Exploration

- Find as many functions as possible whose graphs go through the origin.
- Find as many functions as possible whose output is 5 when the input is 2.