Introduction to Abstract Algebra

Lessons for high school students adapted from
Algebra: Themes, Tools, Concepts
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www.MathEducation.page/abs-alg

1. Math on Another Planet: Small Pockets

On the treeless planet of Glosia, the currency consists of florins, ecus, and ducats. *One florin is worth two ecus, and one ecu is worth two ducats*. Since there is no paper, there is no paper money, and the people of Glosia have to carry coins everywhere. King Evariste VII, being immensely rich, must wear bloomers with enormous reinforced pockets to hold his money.

One day, the King realizes that there is a new trend in Glosian fashion: elegant men and women wear only small pockets. Evariste VII, not one to be left behind by the great movements of style, decides to institute a drastic economic reform, by enacting a strange law: *One ducat is worth two florins!* (The old rules are not changed.) When you realize trades can be made in either direction, you can see how the King's brilliant legislation will abolish poverty forever!

The people of Glosia are ecstatic. With the new system, one may have a fortune in one's pockets, and yet never carry more than three coins! One can be rich and fashionable at the same time. For example, if you own eight ecus, you can go to the bank, and trade them in for four florins. These can be traded again, for two ducats, which equal one ecu, which will certainly fit in your pocket.

1. **Exploration**.
   a. The King trades his coins at the bank, according to their official value, with the object of having as few coins as possible in the tiny pocket of his slinky new pants. He starts with 1000 florins. What does he end up with?
   b. Prince Enbel has one ducat. He buys a toastereo (a popular appliance which, unfortunately, does not make coffee), costing 50 ecus. If he is given the fewest coins possible, how much change does he get?
   c. Princess Lisa has one ecu. She wins the first prize in a contest in *Names Magazine*. The prize is one ducat, one ecu, and one florin. She now has four coins, but they won't fit into her pocket. What does she have after trading them in to get as few coins as possible? (The second prize would have been a T-shirt with the *Names* logo and no pockets at all.)
   d. Sol Grundy has no money. He gets a job at the toastereo store, earning one florin per day, seven days a week. Since his pockets are fashionably small, he trades his money as often as possible in order to have as few coins as possible. If he starts his new job on Monday, how much does he have each day of the week? And the next week? (Assume he doesn't spend any money.)

2. Make a list of the amounts of money one can have that cannot be reduced to a smaller number of coins. (Hint: there are seven possible amounts.) One of the amounts is (d + e).

3. Make an addition table for Glosian money. It should be a seven by seven table, with a row and column for each of the amounts you found in the previous problem. For example, your table should show that (d + e) + d = f.

4. One of the seven amounts you found in the previous problem can be considered to be the *zero* of Glosian money, since adding it to a collection of coins does not change the collection's value (after trading to get the smallest possible number of coins, of course.) Which amount is the zero for Glosian money?

5. The *opposite* of an amount is the amount you add to it to get the zero. Find the opposite of each of the seven amounts.

Glosian Money Addition Table

+	d	e	f	d + e	e + f	d + f	d + e + f
d							
e							
f							
d + e							
e + f							
d + f							
d + e + f							

2. Math on Another Planet: A Long Month

The King can never remember which month this is and how many days the month has. He decides to start a new calendar, with a single infinite month, the month of Evary, named after himself. This is what the calendar looks like:

Evary						
Mo	**Tu**	**We**	**Th**	**Fr**	**Sa**	**Su**
			1	2	3	4
5	6	7	8	9	10	11
12	13	14	15	16	17	18
19	20	21	22	23	24	25
26	27	28	29	30	31	32
33	34	35	36	37	38	…

1. What day of the week will it be on Evary 100th? Explain how you figured it out.

The King is so pleased with the new calendar that he decides to invent a new kind of math. He calls it "Calendar Math". In Calendar Math, Monday + Tuesday $\rightarrow 5 + 6 = 11 \rightarrow$ Sunday, or, more briefly, Mo + Tu = Su.

2. Check whether if you picked different numbers for Monday (such as 12, 19, etc.) and Tuesday (13, 20, etc.) you would still get Sunday for the sum.

3. Make an addition table for Calendar Math:

+	**Mo**	**Tu**	**We**	**Th**	**Fr**	**Sa**	**Su**
Mo		Su					
Tu							
We							
Th							
Fr							
Sa							
Su							

4. *Calendar Zero* is a day of the week such that when you add it to any other day, you get that other day for the answer. What day is Calendar Zero?

5. Find the *Calendar Opposite* for each day of the week. That is the day you add to a given day to get Calendar Zero. If a day does not have an opposite, or is its own opposite, explain.

6. Calculate:
   a. Mo + Mo
   b. Mo + Mo + Mo
   c. Mo + Mo + Mo + Mo
   etc.

7. How many times do you add Mo to itself to get back Mo?

8. Make a multiplication table for Calendar Math. Here is an example of a result that would appear in it:
   Mo · Tu → 5 · 6 = 30 → Fr,
   so: Mo · Tu = Fr

·	**Mo**	**Tu**	**We**	**Th**	**Fr**	**Sa**	**Su**
Mo		Fr					
Tu							
We							
Th							
Fr							
Sa							
Su							

9. What is special about Calendar Zero in multiplication?

10. *Calendar One* is a day of the week such that when you multiply it by any other day, you get that other day for the answer. What day is Calendar One?

11. The *Calendar Reciprocal* of a day is the day you multiply it by to get calendar one. Find the Calendar Reciprocal for each day. If a day does not have a reciprocal, or is its own reciprocal, explain.

12. Calculate Su^2, Su^3, etc. What power of Su is equal to Su?

13. **Summary.** Summarize Calendar Math.

3. Letter Strings

In abstract algebra, letters do not stand for numbers. Abstract algebra has many applications, for example to particle physics, or to the analysis of the Rubik's cube. Here is a simple example.

The YZ Game

In this game, the object is, starting with a string of Ys and Zs, to simplify the string by following strict rules. The rules are:

`YYY` can be erased
`ZZ` can be erased
the commutative law: `YZ = ZY`

Examples:

a.
```
YZZYYZYZYYZ  (erase ZZ)
Y  YYZYZYYZ  (erase YYY)
     ZYZYYZ  (commute YZ)
     ZZYYYZ  (erase ZZ and YYY)
          Z  (can't be simplified)
```

b.
```
ZYYYZ  (erase YYY)
Z   Z  (erase ZZ)
  E    (the empty string is left)
```

1. Simplify the strings:
   a. YZYZZYYZ   b. YYYYZZYZY
   c. YZYZYZYZYZYZYZYZZZYZYZYYZY

Including the empty string E, there are 6 essentially different strings that cannot be simplified. They are called the *elements of the YZ group*.

2. Find all the elements of the YZ group.

The symbol $\leftrightarrow$ represents the operation "put together and simplify". For example:

$\mathrm{Y} \leftrightarrow \mathrm{YY} = \mathrm{E}$
$\mathrm{YZ} \leftrightarrow \mathrm{YZ} = \mathrm{YY}$
$\mathrm{Y} \leftrightarrow \mathrm{E} = \mathrm{Y}$

3. Compute:
   a. $\mathrm{E} \leftrightarrow \mathrm{YZ}$ b. $\mathrm{YZ} \leftrightarrow \mathrm{YY}$ c. $\mathrm{Z} \leftrightarrow \mathrm{YZ}$

4. Find the missing term:
   a. $\mathrm{YZ} \leftrightarrow$ ___ $= \mathrm{E}$ b. $\mathrm{Z} \leftrightarrow$ ___ $= \mathrm{YZ}$ c. $\mathrm{YY} \leftrightarrow$ ___ $= \mathrm{Z}$

For the YZ group, ↔ works a little bit like multiplication. Another way to write the first two rules is: $Y^3 = E$ and $Z^2 = E$

5. The only powers of `Y` are: `Y`, Y^2, and `E`. Explain.

6. Find *all* the powers of each element of the YZ group.

7. Simplify (show your work):
   a. Y^{1000} b. $(YZ)^{1001}$

8. Make a "↔ table".

↔	**E**	**Y**	**YY**	**Z**	**YZ**	**YYZ**
E						
Y						
YY						
Z						
YZ						
YYZ						

9. What element of the group works like 1 for multiplication?

10. What is the reciprocal of each element? (In other words, for each element, what element can be put together with it to get the "1"?)

The yz Game

For this group, the rules are:
`yyy` can be erased
`zz` can be erased
`yzy = z`
The empty string is called `e`.
There is no commutative law.

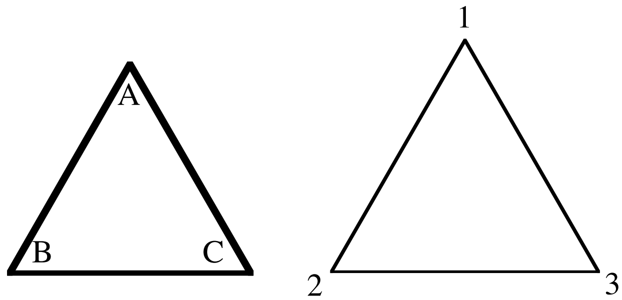
11. * Do Problems 1-10 for the yz group. (Hint: `zyy` and `yyz` *can* be simplified.)

12. **Report:** Write a report on the yz group.

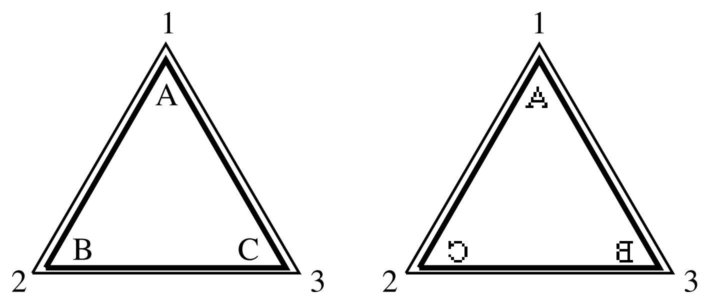
4. Smooth Moves

You will need: a cardboard equilateral triangle

Flips and Turns



1. a. Write the letters A, B, and C on your triangle, near the vertices. Make sure the same letter appears on both sides of the cardboard at each vertex.
   b. Outline the triangle on a piece of paper, and write the numbers 1, 2, and 3 outside the outline, as in the figure.



There are several different ways you can place the triangle on its outline. The two ways shown in the figure can be written ABC and ACB. ABC is called the home position.

You can get from the home position to each position by using one of the following moves (see the figure on the next page.)

Turns:

- the clockwise turn (abbreviation: c)
- the counterclockwise turn (abbreviation: a — short for “anticlockwise”)

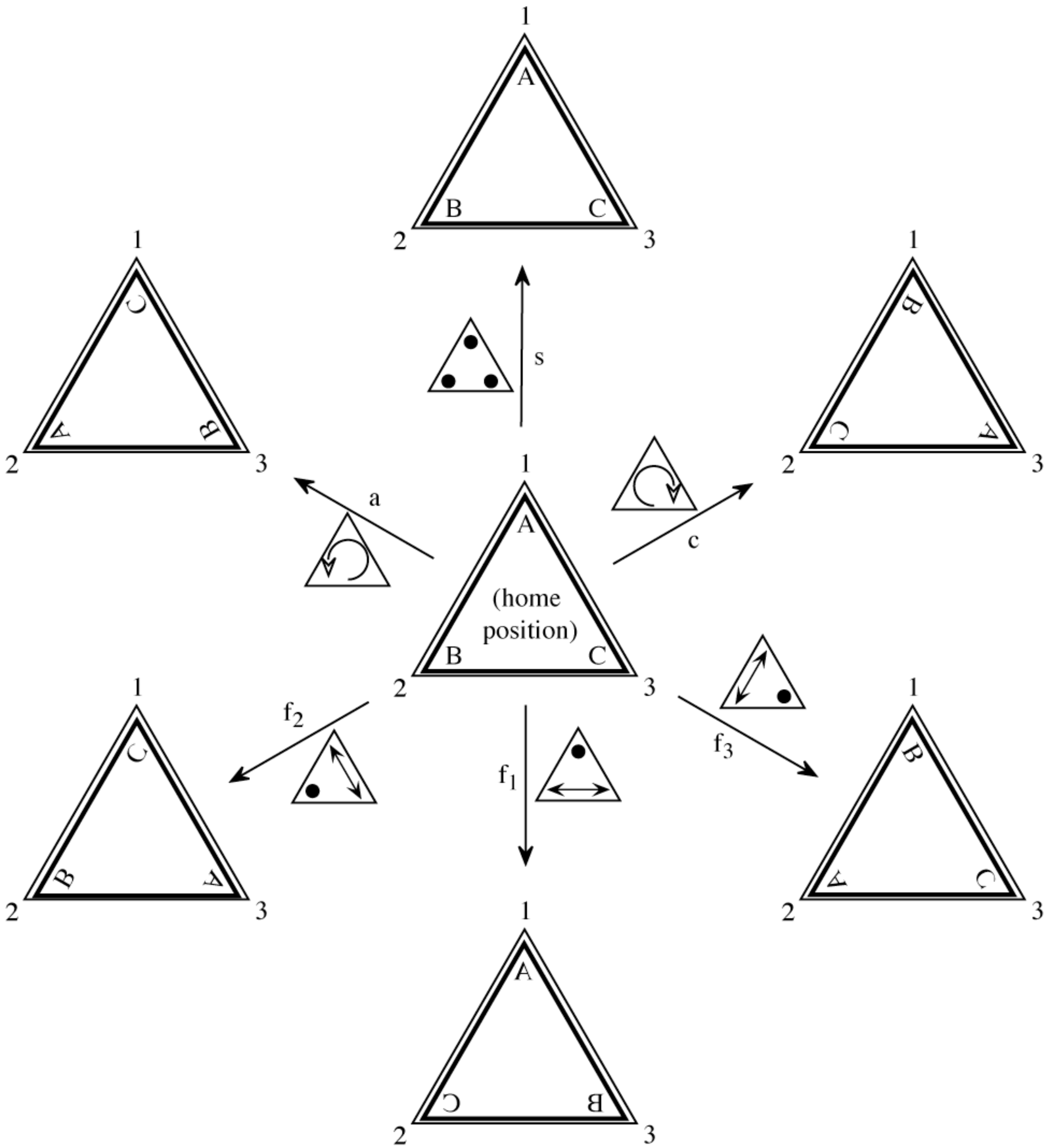
To do the turns (also called *rotations*), you do not lift the triangle off the page. You turn until the triangle fits into the outline again.

Flips:

There are three flips. To do a flip, you keep one corner in place, and have the other two switch positions. For example, for flip 2 (f_2), you keep corner #2 fixed, and corners 1 and 3 switch positions. Flips are also called *reflections*.

Stay:

The “move” that does not move! (s)



(The dot means: “this corner stays in place”.)

2. Which corner stays fixed and which changes position
   a. for flip 3 (f_3)?
   b. for flip 1 (f_1)?

Practice the turns and flips, making sure you know what each one does. In this lesson, you will have to execute turns and flips in succession, without going back to the home position in between.

Example: Do f_1, then a. (Such a sequence is simply written f_1 a.) If you start at the home position, and do these moves in order, you will end up in the position BAC. (Try it.) But since you could have ended up there *in one move* (f_3), you can write: f_1 a = f_3.

3. Find out whether a f_1 = f_3

4. Simplify. That is, give the one move that has the same result as the given sequence of moves.
   a. a a   b. f_1 f_3 c. f_3 f_1
   d. s f_2 e. a c   f. c a

5. Simplify.
   a. f_1 f_2 f_3 b. a f_1 a f_2 a f_3
   c. f_1 a f_2 a f_3 a   d. c f_1 c f_2 c f_3

6. Write each of the six moves in terms of only f_1 and c.

7. Write each of the six moves in terms of only f_1 and f_2.

8. Fill in the blanks:
   a. a ___ = f_1 b. ___ a = f_1
   c. f_1 ___ = f_2 d. ___ f_1 = c

5. The Algebra of Moves

Executing moves in order is an *operation* on triangle moves, just like multiplication is an operation on numbers. The set of six moves, together with this operation, is called the *symmetry group* for the triangle.

1. € Make a "multiplication table" for triangle moves. That is, figure out the one move that has the same result as doing the two given moves. Describe any interesting patterns you find in the finished table.

Then...

First...	**s**	**c**	**a**	**f_1**	**f_2**	**f_3**
s						
c						
a						
f_1			f_3			
f_2						
f_3						

2. € For each of the six moves, what move undoes it?

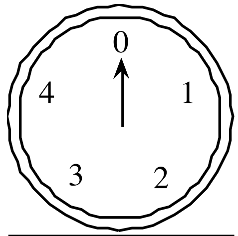
Executing one move (or sequence) repeatedly can be written with *power notation*. For example, $f_2{}^7$ means: execute f_2 seven times.

3. Simplify:
   a. a^{999}
   b. c^{1000}
   c. $f_2{}^{1000}$
   d. $(af_2)^{1001}$

4. **Project.** What flips and turns are possible for another figure, such as a rectangle or a square? Write a report on the symmetry group for that figure.

6. Group Theory

Mod Clocks



Mod 5 Clock

The figure shows a mod clock, which is a special function machine. For any positive whole number input, it will output a number between 0 and 4. For example:

input	output
1	1
9	4
13	3
25	0
77	2

input	output
5	0
12	2
17	2
26	1
100	0

1. What would be the output of the mod clock for the following inputs? Explain.
   a. 1998 b. 1899 c. 9981

Definition: $a \oplus b$ is the output from the mod clock for the input a+b. $a \otimes b$ is the output for the input ab.

Example: $3 \oplus 2 = 0$, and $3 \otimes 2 = 1$

2. Make a table for each of $\oplus$ and $\otimes$:

$\oplus$	0	1	2	3	4
0					
1					
2					
3					
4					

$\otimes$	0	1	2	3	4
0					
1					
2					
3					
4					

3. **Generalization.** The clock above is a mod 5 clock. Find ways to predict the output of mod 10, mod 2, mod 9, and mod 3 clocks.

Groups

Definition: a *group* is a set of elements, together with an operation that satisfies the following rules.

- *closure:* using the operation on two elements of the group yields an element of the group.
- *associative law:* (a b) c = a (b c).
- *identity element:* one of the elements, e, is such that a e = e a = a, for any element a in the group.
- *inverse element:* every element a has an inverse a' such that a a' = a' a = e

Some groups are *commutative* (a b = b a) and some are not.

4. a. Show that the set {0,1,2,3,4} together with the operation ⊕ is a group
   b. Show that {0,1,2,3,4} with ⊗ is not a group
   c. Show that {1,2,3,4} with ⊗ is a group

5. Is the set of the integers a group with the following operations?
   a. addition
   b. multiplication

6. Show that the set of rational numbers (positive and negative fractions) together with multiplication is not a group. By removing one element, it can be made into a group. Which element? Explain.

7. Think about a mod 4 clock, with the numbers {0,1,2,3}. Is it a group for ⊕? ⊗? Can it be made into one by removing an element?

8. **Report.** Give examples of groups. For each one, give the set and the operation, and explain how they satisfy the rules. Include finite and infinite groups, commutative and non-commutative group.